AN EXISTENCE THEOREM FOR A DIFFERENTIAL EQUATION OF THE SECOND ORDER, WITH AN APPLICATION TO THE CALCULUS OF VARIATIONS*

BY

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PICARD has shown † that the equation

$$\eta'' = f(\xi, \eta, \eta') \ddagger \tag{1}$$

has a solution $\eta = \eta(\xi)$ joining two given points p and q, provided that f satisfies certain continuity restrictions and that q lies in a properly chosen region in the vicinity of the point p. The constants of integration are then the coördinates of p and q. In the following pages it is proposed to discuss the character of the solution as a function of these coördinates, and to extend PICARD'S theorem to the equation

$$\frac{1}{\rho} = G(x, y, x', y'), \tag{2}$$

where the solution is sought in the form

$$x = x(t), \qquad y = y(t), \tag{3}$$

and where

$$\frac{1}{\rho} = \frac{x'y'' - x''y'}{(x'^2 + y'^2)^{\frac{3}{2}}}.$$

In § 3 an application is made to the problem of minimizing the integral

$$I = \int_{t_0}^{t_1} F(x, y, x', y')\, dt$$

taken along a curve in the form (3). It is shown that a finite closed region S of the (x, y)-plane and a constant $\delta > 0$ can be found, such that any two points

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† *Traité d'Analyse*, vol. 3, pp. 94–100.

‡ Primes indicate differentiation with respect to the variables ξ, t, τ, s, which in the following pages do not occur in the same functions.

of S whose distance is less than δ can always be joined by a continuous extremal of the problem with a continuously turning tangent, and this extremal minimizes the integral I.

In his lectures* on the calculus of variations WEIERSTRASS used the first part of this result for the purpose of constructing a field in the neighborhood of a given point. The theorem also plays an essential role in the recent existence proofs of HILBERT† and NOBLE‡ in the calculus of variations. So far as is known to the writer, however, no proofs have been given. In the proof of § 3 HILBERT's notion of the invariant integral has been extended to the problem of the calculus of variations for curves in parametric representation.

§ 1. *The character of Picard's solution.*

Suppose that the function f in equation (1) is continuous with continuous first partial derivatives f_η, $f_{\eta'}$§ for all values of ξ, η, η' in a region

$$R: \qquad |\xi| < 2a, \qquad |\eta| < 2b, \qquad |\eta'| < 2b',$$

and let M, α, β be positive constants greater than the absolute values of f, f_η, $f_{\eta'}$ respectively for such arguments. If $b' < b_0' < 2b'$, and a region R_0 is defined entirely within R by the inequalities

$$R_0: \qquad |\xi| \leqq a, \qquad |\eta| \leqq b, \qquad |\eta'| \leqq b_0', \tag{4}$$

then a function

$$\eta = \phi(\xi, \xi_0, \eta_0, \eta_0'), \tag{5}$$

and a positive constant l‖ exist with the following properties:

1) ϕ, ϕ', ϕ'' are single-valued, continuous, and have continuous first partial derivatives with respect to ξ_0, η_0, η_0' for all points (ξ_0, η_0, η_0') in R_0, and for $|\xi - \xi_0| < l$;

2) $\eta = \phi$ is a solution of the equation (1), such that η, η' have the initial values η_0, η_0' for $\xi = \xi_0$, and (ξ, η, η') lies in R;

3) η is the only continuous function of ξ with continuous derivative on the interval $|\xi - \xi_0| < l$, which has the properties described in 2).¶

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*Summer, 1879.

†HILBERT, *Lectures on the calculus of variations*, 1900. HILBERT extends the definition of the definite integral of the calculus of variations, making it apply to a class of functions without derivatives. The identity of this definition with one given previously by WEIERSTRASS has been proved by OSGOOD by means of the theorem which is the subject of this paper. See the Transactions of the American Mathematical Society, vol. 2 (1901), p. 294, footnote.

‡NOBLE, *Eine neue Methode in der Variationsrechnung*, Dissertation, Göttingen, 1901.

§Literal subscripts denote partial differentiation.

‖ l is the smallest of the quantities a, b/M, $2b' - b_0'/M$.

¶For the existence of the function (5), see, e. g., PICARD, *Traité d' Analyse*, vol. 2, chap. 11; for the uniqueness, and existence of derivatives with respect to the constants, see especially PEANO, Atti di Torino, vol. 33 (1897), p. 9. Other references are given by PAINLEVÉ,

Consider now another point (ξ_1, η_1) besides (ξ_0, η_0). If η_0' can be so determined that

$$\eta_1 = \phi(\xi_1, \xi_0, \eta_0, \eta_0'), \tag{6}$$

then the corresponding solution (5) will pass through both points. If furthermore $\partial\phi(\xi_1, \xi_0, \eta_0, \eta_0')/\partial\eta_0' \neq 0$, then equation (6) can be solved for η_0' as a function of $\xi_0, \eta_0, \xi_1, \eta_1$, and solutions of (1) will be determined which pass through any two points in the neighborhoods of the two originally considered.

From the results of PICARD * it follows that when (ξ_1, η_1) is properly chosen, there is one and only one solution η_0' of (6). The region about (ξ_0, η_0)

$$R_1: \qquad |\xi - \xi_0| < a, \qquad |\eta - \eta_0| < b, \qquad |\eta'| < b_0',$$

is entirely within R, and consequently the assumptions for f still hold true. PICARD has shown that if a positive constant δ satisfies the inequalities †

$$(\Delta_1) \quad \delta < l, \qquad M\frac{\delta^2}{8} + \delta b' < b, \qquad M\frac{\delta}{2} + b' < b_0', \qquad \alpha\frac{\delta^2}{8} + \beta\frac{\delta}{2} < 1,$$

and if furthermore (ξ_1, η_1) lies in the V-shaped region about (ξ_0, η_0),

$$0 < \xi_1 - \xi_0 \leqq \delta, \qquad |\eta_1 - \eta_0| \leqq (\xi_1 - \xi_0)b', \tag{7}$$

then there exists a continuous solution of (1) with a continuous derivative on the interval $\xi_0 \leqq \xi \leqq \xi_1$, which joins (ξ_0, η_0) with (ξ_1, η_1). This solution lies entirely within R_1, and is the only one with the given properties even in R. The initial value η_0' so defined satisfies the inequalities (4) and is the desired solution of equation (6).

Furthermore, $u = \partial\phi(\xi, \xi_0, \eta_0, \eta_0')/\partial\eta_0'$ cannot vanish for such a value of η_0' when $\xi = \xi_1$. It is, namely, a solution of the equation

$$u'' = uf_\eta + u'f'_\eta, \tag{8}$$

which is not identically zero because at $\xi = \xi_0$ the initial values of η, η' are η_0, η_0', and consequently

$$0 = \frac{\partial\phi}{\partial\eta_0'}, \qquad 1 = \frac{\partial\phi'}{\partial\eta_0'}.$$

But according to PICARD ‡ no solution of (8) except $u \equiv 0$ vanishes twice on the interval $\xi_0 \leqq \xi \leqq \xi_1$.

Encyclopädie der Mathematischen Wissenschaften, IIA4a, pp. 195, 200. The theorem is true if R_0 is any region entirely interior to R, but the value of l must be chosen differently.

* *Traité d'Analyse*, vol. 3, pp. 94–100.

† The form of statement here given differs slightly from PICARD's.

‡ Loc. cit. p. 100. It is important to notice that the PICARD coefficients a_{ik}, b_{ik} are continuous, and that the interval on which the existence proof can be made for his equations (E), is the same as the interval for the original equations.

It follows then from the foregoing and from the theorems on implicit functions* that (6) can be solved for η_0', and that the resulting function of $\xi_0, \eta_0, \xi_1, \eta_1$ is single-valued, continuous, and has continuous first partial derivatives for all values of ξ_0, η_0 which satisfy (4), and of ξ_1, η_1 which satisfy (7). After substitution of this function in (5), the latter takes the form

$$\eta = \psi(\xi, \xi_0, \eta_0, \xi_1, \eta_1), \tag{9}$$

where ψ has the following properties:

1) ψ, ψ', ψ'' are single-valued and continuous, and have continuous first partial derivatives with respect to $\xi_0, \eta_0, \xi_1, \eta_1$ for all values of the arguments satisfying the conditions,

$$|\xi_0| \leqq a, \qquad |\eta_0| \leqq b,$$

$$0 < \xi_1 - \xi_0 \leqq \delta, \qquad |\eta_1 - \eta_0| \leqq (\xi_1 - \xi_0)b',$$

$$\xi_0 \leqq \xi \leqq \xi_1;$$

2) $\eta = \psi$ solves equation (1), has the initial and end-values η_0, η_1 at ξ_0 and ξ_1, and the points (ξ, η, η') defined are all in R;

3) η is the only continuous function of ξ with a continuous derivative on the interval $\xi_0 \leqq \xi \leqq \xi_1$, which has the properties described in 2).

§ 2. *Extension of the preceding results.*

In order to extend the results of § 1 to the equation (2), suppose

1) G is continuous with continuous first partial derivatives when (x, y) lies in a certain finite closed region S_1 of the (x, y)-plane, and x', y' are any values not both zero;

2) $$G(x, y, \kappa x', \kappa y') = G(x, y, x', y') \qquad (\kappa > 0),\dagger$$

for all points (x, y, x', y') described in 1).

If S is a closed region entirely interior to S_1, a positive quantity δ can be found such that any two points $P(x_0, y_0)$ and $Q(x_1, y_1)$ of S, whose distance PQ is less than δ, can be joined by a continuous solution of (2) with a continuously turning tangent. To prove this, the (x, y)-plane can be transformed into a (ξ, η)-plane by translation and rotation, so that P goes into the new origin and the line PQ into the new ξ-axis. Equation (2) becomes

$$\frac{1}{\rho} = \Gamma(\xi, \eta, \xi', \eta'), \tag{10}$$

* See, for example, JORDAN, *Cours d'Analyse*, 2d ed., vol. 1, p. 80.

† This is the necessary and sufficient condition that G, like ρ, should be independent of the parametric representation. For a similar case, see KNESER, *Lehrbuch der Variationsrechnung*, p. 6.

where Γ has the properties 1) and 2) with respect to a region Σ_1, the transform of S_1. On account of 2), ξ can be choosen as the independent variable, so that (10) takes the form

$$\eta'' = (1+\eta'^2)^{\frac{3}{2}}\Gamma(\xi, \eta, 1, \eta') = f(\xi, \eta, \eta'). \tag{11}$$

Since S is entirely interior to S_1, a positive constant ϵ exists such that a circle of radius ϵ about any point of S lies also entirely within S_1. A region R for the equation (11) is then defined by selecting b' arbitrarily and taking

$$2a = 2b = \frac{\epsilon}{\sqrt{2}}. \tag{12}$$

The values of the constants M, α, β are independent of the choice of P and Q in S, and can be determined as follows. The absolute values of G and its derivatives have a maximum N when (x, y) lies in S_1 and $x'^2 + y'^2 = 1$. The corresponding maxima for Γ, Γ_η, $\Gamma_{\eta'}$ are easily seen to be not greater than N, $2N$, $2N$ respectively. On account of 2) the absolute values of Γ, Γ_η do not exceed these constants when ξ', η' take any values whatsoever which are not both zero. But

$$\Gamma_{\eta'}(\xi, \eta, \kappa\xi', \kappa\eta') = \frac{1}{\kappa}\Gamma_{\eta'}(\xi, \eta, \xi', \eta'),$$

and therefore

$$|\Gamma_{\eta'}(\xi, \eta, \xi', \eta')| < \frac{2N}{\sqrt{\xi'^2 + \eta'^2}} \tag{13}$$

whatever $(\xi', \eta') \neq (0, 0)$ may be. The values of M, α, β are obtained from the following inequalities, derived with the help of (13):

$$|f| < (1+4b'^2)^{\frac{3}{2}}N,$$

$$|f_\eta| = |(1+\eta'^2)^{\frac{3}{2}}\Gamma_\eta| < (1+4b'^2)^{\frac{3}{2}} \cdot 2N,$$

$$|f_{\eta'}| = |\tfrac{3}{2}(1+\eta'^2)^{\frac{1}{2}}\Gamma + (1+\eta'^2)^{\frac{3}{2}}\Gamma_{\eta'}| < \tfrac{3}{2}(1+4b'^2)^{\frac{1}{2}}N + (1+4b'^2)2N.$$

The following notations will be used;

s = the length of arc measured from P along a solution (3) passing through P;

τ = the angle between the positive x-axis and the direction of the tangent for increasing s;

r = the length of the radius vector from P to the point of the solution corresponding to s;

ϑ = the angle between the positive x-axis and the radius vector.

Suppose now $\delta > 0$ determined as in § 1 for the equation (11), and that the distance from P to Q is less than δ. Then it follows that there exists one and

only one continuous solution of the equation (2) with a continuously turning tangent, joining P and Q and satisfying the inequalities

$$|r\sin(\vartheta-\vartheta_1)|<2b, \qquad |\tan(\tau-\vartheta_1)|<2b',$$

where ϑ_1 is the value of ϑ at Q. These inequalities, which are derived from the ones defining R, are equivalent to

$$|r\sin(\vartheta-\vartheta_1)|<2b, \qquad |\sin(\tau-\vartheta_1)|<\frac{2b'}{\sqrt{1+4b'^2}}. \tag{14}$$

The constant δ can be further restricted, however, so that *every* continuous solution of (2) with a continuously turning tangent, which joins P and Q and lies entirely within a circle of radius δ about P, must satisfy the inequalities (14). Consequently there can be but one such solution in the whole circle. The former of the inequalities (14) is easily seen to be true, for from (Δ_1) and (12)

$$|r\sin(\vartheta-\vartheta_1)|<\delta<a=b.$$

In order to make the second inequality hold true, the equations of the curve (3) can be written in the form *

$$x-x_0=\int_0^s\cos\tau'ds', \qquad y-y_0=\int_0^s\sin\tau'ds', \tag{15}$$

the primes distinguishing the variables of the integrand. Then

$$\frac{d\tau}{ds}=\frac{1}{\rho}=G\left(x,y,\frac{dx}{ds},\frac{dy}{ds}\right), \tag{16}$$

and the square of the radius vector has the second derivative

$$\begin{aligned}\frac{d^2r^2}{ds^2}&=2\left\{1+(x-x_0)\frac{d^2x}{ds^2}+(y-y_0)\frac{d^2y}{ds^2}\right\}\\&=2\left\{1-G\left(\sin\tau\int_0^s\cos\tau'ds'-\cos\tau\int_0^s\sin\tau'ds'\right)\right\}\\&=2\left\{1-G\int_0^s\sin(\tau-\tau')ds'\right\}\\&\geqq 2\{1-Ns\}.\end{aligned}$$

Since dr^2/ds vanishes for $s=0$, the last inequality shows that dr^2/ds and therefore r constantly increase with s, when s remains on the interval

$$0\leqq s\leqq\frac{1}{N}. \tag{17}$$

* Encyclopädie der Mathematischen Wissenschaften, III D 1, 2, p. 34.

After two integrations from 0 to s and a root extraction, it follows that

$$r \geqq s\sqrt{1 - \tfrac{1}{3}Ns} \geqq s\sqrt{\tfrac{2}{3}}. \tag{18}$$

Consequently r attains the value $\sqrt{\frac{2}{3}}/N$ while s is still on the interval (17). If therefore

$$\delta < \frac{1}{N}\sqrt{\tfrac{2}{3}} \tag{Δ_2}$$

and the arc PQ is entirely within the circle of radius δ about P, then s cannot exceed $1/N$ and the inequality (18) holds true.

From (16) it follows that

$$|\tau - \tau'| = \left|\int_{s'}^{s} G\,ds\right| \leqq N\,|s - s'|, \tag{19}$$

where τ' is the value of τ at any point ($s = s'$) of the arc PQ. Further, from (15),

$$\sin(\tau - \vartheta_1) = \sin\tau \cdot \frac{x_1 - x_0}{r_1} - \cos\tau \cdot \frac{y_1 - y_0}{r_1} = \frac{1}{r_1}\int_0^{s_1} \sin(\tau - \tau')\,ds',$$

where r_1, s_1 are the values of r, s at Q. Consequently, from (19) and (18),

$$|\sin(\tau - \vartheta_1)| \leqq \frac{1}{r_1}\int_0^{s_1} |\tau - \tau'|\,ds' \leqq N\frac{s_1^2}{2r_1} \leqq \tfrac{3}{4}Nr_1,$$

and the second of the inequalities (14) will be satisfied if

$$\tfrac{3}{4}N\delta < \frac{2b'}{\sqrt{1 + 4b'^2}}. \tag{Δ_3}$$

It has been proved therefore that there is one and but one continuous solution of (2) with a continuously turning tangent, which joins P and Q and lies entirely within a circle of radius δ about P, provided that δ satisfies (Δ_1), (Δ_2) and (Δ_3). If the length s along the arc PQ is taken as parameter, the equations of this solution can easily be found. The relations between the (x, y)- and (ξ, η)-coördinates are

$$x = x_0 + \xi\cos\vartheta_1 - \eta\sin\vartheta_1, \qquad y = y_0 + \xi\sin\vartheta_1 + \eta\cos\vartheta_1, \tag{20}$$

and the relation between ξ and s is

$$\int_{\xi_0}^{\xi} \sqrt{1 + \eta'^2}\,d\xi = s. \tag{21}$$

On account of the theorems on implicit functions and the properties of (9), the equation (21) has a solution

$$\xi = \xi(s, \xi_0, \eta_0, \xi_1, \eta_1), \tag{22}$$

where the function ξ, together with its derivatives ξ', ξ'', is continuous and has continuous first partial derivatives with respect to ξ_0, η_0, ξ_1, η_1 for all points (ξ_0, η_0) and (ξ_1, η_1) within small circles about $(0, 0)$ and $(r_1, 0)$ respectively, and for

$$0 \leqq s \leqq s_1 = \int_{\xi_0}^{\xi_1} \sqrt{1 + \eta'^2}\, d\xi. \tag{23}$$

This function ξ can now be substituted in (9), and then both (22) and (9) in (20). The resulting equations give all the solutions of (2) joining points in the neighborhood of P and Q. If ξ_0, η_0, ξ_1, η_1 are replaced by their values in (x, y)-coördinates, it follows that *there exist two functions*

$$x = g(s, x_0, y_0, x_1, y_1), \qquad y = h(s, x_0, y_0, x_1, y_1) \tag{24}$$

with the properties:

1) *g', g'', h', h'' exist and with g, h are single-valued and continuous, and have continuous first partial derivatives with respect to x_0, y_0, x_1, y_1 for all values of the five arguments such that*

a) (x_0, y_0) *is in* S, *and* $\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} < \delta$,

b) $0 \leqq s \leqq s_1$;*

2) *the curve defined by* (24) *lies entirely in a circle of radius* δ *about* (x_0, y_0), *solves equation* (2), *and passes through* (x_0, y_0), (x_1, y_1) *for* $s = 0$ *and* $s = s_1$ *respectively;*

3) *this is the only continuous curve with continuously turning tangent which has the properties described in* 2).

§ 3. *Application to the calculus of variations.*

In the integral I suppose that the function F is continuous with continuous first, second and third partial derivatives, and that

$$F(x, y, \kappa x', \kappa y') = \kappa F(x, y, x', y') \qquad (\kappa > 0), \tag{25}$$†

$$F_1 = \frac{1}{y'^2} F_{x'x'} = -\frac{1}{x'y'} F_{x'y'} = \frac{1}{x'^2} F_{y'y'} > 0, \tag{26}$$‡

* From equation (23) it follows that s_1 is a continuous function of x_0, y_0, x_1, y_1 with continuous first partial derivatives.

† This is the condition for the integral I to be invariant under change of parametric representation. See KNESER, loc. cit.

‡ For the possibility of defining F_1 in this way see KNESER, loc. cit. p. 52. Problems for which $F_1 > 0$ have been named *regular* by HILBERT; *Mathematical Problems*, Bulletin of the American Mathematical Society, ser. 2, vol. 8 (1902), p. 469.

for all points (x, y) in a finite closed region S_1 of the (x, y)-plane, and for any values of x', y' which are not both zero. EULER's equation for I is

$$F_{xy'} - F_{x'y} + F_1(x'y'' - x''y') = 0,^* \tag{27}$$

or

$$\frac{1}{\rho} = \frac{F_{x'y} - F_{xy'}}{(x'^2 + y'^2)^{\frac{3}{2}} F_1} = G(x, y, x', y'),$$

which has the properties of equation (2) in the region S_1. Any two points $P(x_0, y_0)$ and $R(x_2, y_2)$ in a region S, and such that the distance PR is less than δ, can therefore be joined by an extremal $\mathfrak{E}$, one of the set (24). Let $\mathfrak{C}$ denote any other continuous curve of the form (3) joining P and R, and consisting of a finite number of pieces, each of which has a continuously turning tangent. Then it is desired to prove that $I_{\mathfrak{E}}$, the value of I taken along $\mathfrak{E}$ from P to R, is smaller than the value $I_{\mathfrak{C}}$ taken along $\mathfrak{C}$.

Suppose that the curve $\mathfrak{C}$ passes through P for $t = t_0$ and through R for $t = t_2$, and in the integral

$$I^*_{\mathfrak{C}} = \int_{t_0}^{t_2} \{x' F_y(x, y, p, q) + y' F_q(x, y, p, q)\}\, dt,^\dagger$$

let p and q be functions of x, y defined by the equations

$$p(x, y) = g'(s_1, x_0, y_0, x, y), \qquad q(x, y) = h'(s_1, x_0, y_0, x, y),$$

in which x_0 and y_0 are kept constant and x_1, y_1 are replaced by x, y. Then the value $I^*_{\mathfrak{C}}$ is independent of the curve $\mathfrak{C}$. For if U, V represent the coefficients of x', y' in the integrand respectively, the criterion for independence of path is that

$$\frac{\partial U}{\partial y} \equiv \frac{\partial V}{\partial x}$$

over the whole circle of radius δ about P.‡ But

$$\frac{\partial U}{\partial y} = F_{py} + F_{pp} p_y + F_{pq} q_y = F_{py} + F_1 q(q p_y - p q_y),$$

$$\frac{\partial V}{\partial x} = F_{qx} + F_{qp} p_x + F_{qq} q_x = F_{qx} + F_1 p(-q p_x + p q_x)$$

* WEIERSTRASS, *Lectures*, 1879.

† This is HILBERT's invariant integral for the problem in parametric representation. Compare with OSGOOD, Annals of Mathematics, ser. 2, vol. 2 (1901), p. 121. The expressions F_p, F_q, etc., will be used to denote partial derivatives of F with the arguments p, q instead of x', y'.

‡ PICARD, *Traité d'Analyse*, vol. 1, p. 85.

on account of (26), and therefore

$$(28)\qquad \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} \equiv F_{qx} - F_{py} + F_1\{p(q_x p + q_y q) - q(p_x p + p_y q)\}.$$

The differential equations

$$\frac{dx}{ds} = p(x, y), \qquad \frac{dy}{ds} = q(x, y)$$

are satisfied by all the solutions (24) which pass through the point P, so that

$$\frac{d^2x}{ds^2} = p_x p + p_y q, \qquad \frac{d^2y}{ds^2} = q_x p + q_y q.$$

The expression (28) is the left hand side of (27) with x and y replaced by the functions (24) in which, furthermore, $s = s_1$. It vanishes identically in x and y.

Consider the difference

$$\Delta I = I_{\mathfrak{E}} - I_{\mathfrak{C}}.$$

On account of (25)

$$F(x, y, x', y') = x' F_{x'} + y' F_{y'},$$

and therefore

$$I_{\mathfrak{C}} = \int_{s_0}^{s_1} \{F_p \cdot p + F_q \cdot q\}\, ds = I_{\mathfrak{C}}^* = I_{\mathfrak{E}}^*.$$

It follows then that

$$\Delta I = I_{\mathfrak{E}} - I_{\mathfrak{C}}^* = \int_{t_0}^{t_1} E(x, y, p, q, x', y')\, dt,$$

where p, q and x', y' determine the directions of $\mathfrak{E}$ and $\mathfrak{C}$ respectively, and

$$E = F(x, y, x', y') - x' F_p - y' F_q.$$

According to WEIERSTRASS,*

$$E = (1 - \cos l) F_1[x, y, \cos(\lambda + l_1), \sin(\lambda + l_1)],$$

where

$$\cos\lambda = p, \qquad \sin\lambda = q,$$

$$\cos(\lambda + l) = \frac{x'}{\sqrt{x'^2 + y'^2}}, \qquad \sin(\lambda + l) = \frac{y'}{\sqrt{x'^2 + y'^2}},$$

and $0 \leqq l_1 \leqq l$. The function E is consequently positive at all points where $l \neq 0$, and therefore $\Delta I > 0$ unless $l \equiv 0$ along the curve $\mathfrak{C}$.

To prove that $l \not\equiv 0$ unless $\mathfrak{C}$ and $\mathfrak{E}$ coincide, suppose the contrary. Then

* *Lectures on the calculus of variations*, 1882.

$$\frac{x'}{\sqrt{x'^2+y'^2}}=p(x,y), \qquad \frac{y'}{\sqrt{p'^2+q'^2}}=q(x,y)$$

along the whole arc PQ of $\mathfrak{C}$. It follows from this equation that $\mathfrak{C}$ has a continuously turning tangent and solves equation (27). The curvature of $\mathfrak{C}$ is, namely, from the last two equations,

$$\frac{1}{\rho}=p(q_x p+q_y q)-q(p_x p+p_y q).$$

Therefore $\mathfrak{C}$ and $\mathfrak{E}$ must coincide on account of the uniqueness of the solution determined by P and Q.

The following theorem has been proved:

The function F in the integral I is supposed to have the properties described at the beginning of § 3, *when (x, y) is in a finite closed region S_1 of the (x, y)-plane and for all values of x', y' which are not both zero. S is a closed region interior to S_1.*

Then a positive quantity δ can be determined such that any two points P and R of S, whose distance is less than δ, can be joined by one and only one continuous solution of Euler's equation which has a continuously turning tangent and lies entirely in a circle of radius δ about the point P. This solution gives to I a smaller value than any other continuous curve which 1) *joins P and R,* 2) *lies in the circle, and* 3) *consists of a finite number of pieces, each of which has a continuously turning tangent.*

If the hypothesis, $F>0$ for all values of x, y, x', y' such that (x, y) is in S_1 and x', y' are not both zero, is added, then a still stronger statement can be made concerning the extremal PQ. Each one of the extremals (24) has a direction τ_0 at the point P and the set can therefore be represented in the form

$$x=\bar{g}(s, x_0, y_0, \tau_0), \qquad y=\bar{h}(s, x_0, y_0, \tau_0). \tag{29}$$

By reasoning similar to the preceding for equations (24), it can be shown from the results in § 1 that these functions, together with $\bar{g}'$, $\bar{h}'$, $\bar{g}''$, $\bar{h}''$, are continuous and have continuous first partial derivatives with respect to x_0, y_0, τ_0 for all values of (x_0, y_0) in the region S, of τ_0 such that

$$-\pi \leqq \tau_0 \leqq +\pi, \tag{30}$$

and of s corresponding to points in the circle of radius δ about P.

Suppose $m>0$ to be the minimum of F when (x, y) is in the closed region S_1 and $x'^2+y'^2=1$. Points on the circumference of the circle of radius δ are defined by values $s \geqq \delta$, and consequently the value of I taken along an extremal from P to the circumference is $\geqq m\delta$. It follows that if i satisfies

$$0 < i \leqq m\delta, \tag{31}$$

then σ can be so determined for each value of τ_0 that

$$\int_0^\sigma F(\bar{y}, \bar{h}, \bar{y}', \bar{h}')ds = i. \tag{32}$$

The function $\sigma(\tau_0, i)$ so defined is single-valued, continuous, and has continuous first partial derivatives for all values of τ_0 and of i on the intervals (30) and (31) respectively. The curve

$$x = \bar{y}(\sigma, x_0, y_0, \tau_0) = x(\tau_0), \qquad y = \bar{h}(\sigma, x_0, y_0, \tau_0) = y(\tau_0)$$

is a closed continuous curve $\mathfrak{C}_i$* about the point P and has a continuously turning tangent. Furthermore the extremal joining P with any other point interior to $\mathfrak{C}_i$ gives to I a value less than i. *If R is a point within the curve $\mathfrak{C}_{m\delta}$, then the extremal PR gives I a smaller value than that given by any other curve $\mathfrak{C}$ whatsoever joining P and R in S_1.* For the extremal minimizes I with respect to all curves joining P and R and lying in the circle of radius δ about P. Any other curve $\mathfrak{C}$ which crosses the circle must also intersect the curve $\mathfrak{C}_{m\delta}$, and at the point of intersection the value of I will already be as large as $m\delta$. Since $F > 0$, it follows that the value of I taken along the whole arc from P to R must be still greater.

It remains to show that *a positive constant δ_1 can be found such that a circle of radius δ_1 about any point P of the region S lies within the corresponding curve $\mathfrak{C}_{m\delta}$*; that is to say, the minimizing property holds uniformly over the whole region S. This follows because from (18) and (32) the radius vector of any point on $\mathfrak{C}_{m\delta}$ satisfies the inequalities

$$r \geqq \sqrt{\tfrac{2}{3}}\,\sigma(\tau_0, m\delta) \geqq \sqrt{\tfrac{2}{3}}\,\frac{m}{M}\,\delta,$$

where M is the maximum of F in the region S_1.

An important illustration of the preceding results is furnished by the geodesic coördinates upon a surface. In this case the integral is

$$I = \int_{t_0}^{t_1} \sqrt{E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2}\,dt,$$

where the surface is taken in the form

$$x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v), \tag{33}$$

* These curves are the so-called "transversals," KNESER, loc. cit., p. 32.

and E, F, G have the usual meaning. If in the (u, v)-region considered the surface has no singularities, and the functions (33) have continuous first and second partial derivatives, then the function under the integral sign satisfies all the hypotheses of § 3, and also $F > 0$. The set of extremals (29) is the totality of geodesic lines through the point P; the curves $\mathfrak{C}_i$ are the geodesic circles about that point; and the geodesic coördinates are τ_0, i.

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DETERMINATION OF ALL THE SUBGROUPS OF THE KNOWN SIMPLE GROUP OF ORDER 25920*

BY

LEONARD EUGENE DICKSON

Introduction.

The trisection of the periods of hyperelliptic functions of four periods, the determination of the 27 lines on a general cubic surface, and the reduction of a binary sextic to the canonical form $T^2 - U^3$, although apparently unrelated, are not essentially distinct problems from the standpoint of group-theory,† since each is readily reduced to the solution of an algebraic equation whose Galois group is the same simple group of order 25920. This equation has been shown to possess resolvents of degrees 27, 36, 40 (two essentially distinct ones), and 45, but none of degree < 27. The last result was established by JORDAN† by an elaborate discussion based on GALOIS's theory of algebraic equations. This result is reëstablished in the present paper, which employs only pure group-theory. All the results mentioned follow from the fundamental theorem (not stated heretofore) *that all the maximal subgroups of the simple* G_{25920} *are conjugate with* G_{960}, G_{720}, G_{648}, H_{648}, *or* G_{576} (§ 70). These five groups, appearing in different notations, play a fundamental rôle in the memoirs of WITTING and BURKHARDT on the geometric and function-theoretic phases of the subject.

Not only in the applications, but also in the theory of groups, the known simple group of order 25920 is of frequent occurrence. In the papers by

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† JORDAN, *Traité des substitutions* (1870), pp. 316–329, 365–369, 666–667; Comptes Rendus (1870), pp. 326–328, 1028; KLEIN, Journal de Mathématiques, ser. 4, vol. 4 (1888), pp. 169–176; WITTING, Mathematische Annalen, vol. 29 (1887), pp. 157–170; MASCHKE, *ibid.*, vol. 33 (1889), pp. 317–344; BURKHARDT, *ibid.*, vol. 35 (1890), pp. 198–296; vol. 38 (1891), pp. 161–224; vol. 41 (1893), pp. 313–343; DICKSON, Comptes Rendus, vol. 128 (1899), pp. 873–5; *Linear Groups* (1901), pp. 303–7.

‡ *Traité*, pp. 319–329. As only typical cases are there treated, much is left for the reader to supply; the case $d = 9$ is not mentioned.

WITTING, MASCHKE, KLEIN, and BURKHARDT, it appears as a quaternary, as a quinary, and as a senary group of linear substitutions with numerical coefficients. Furthermore, it appears in the papers by JORDAN and the writer* as a quaternary abelian group G_{25920} modulo 3, as a quinary orthogonal group O_{25920} modulo 3, as a senary hypoabelian group modulo 2, and as a quaternary hyperabelian group in the $GF[2^2]$. Two sets of generational relations for it have been given, with respectively G_{720} and G_{960} in the foreground.*

Use is made of the results in the following papers by the writer:

I. *Canonical forms of quaternary abelian substitutions in an arbitrary Galois field*, Transactions, vol. 2 (1901), pp. 103–138.

II. *On the subgroups of order a power of p in the quaternary abelian group in the Galois field of order p^n*, Transactions, vol. 4 (1903), pp. 371–386.

III. *The subgroups of order a power of 2 of the simple quinary orthogonal group in the Galois field of order $p^n = 8l \pm 3$*, Transactions, vol. 5 (1904), pp. 1–38.

IV. *Determination of all groups of binary linear substitutions with integral coefficients taken modulo 3 and of determinant unity*, Annals of Mathematics, second series, vol. 5 (1903–4).

V. *Two systems of subgroups of the quaternary abelian group in a general Galois field*, Bulletin of the American Mathematical Society, second series, vol. 10 (1904), pp. 178–184.

To these reference will be made by the corresponding Roman numeral with a subscript to indicate the page. Thus III_8 denotes page 8 of the third paper.

In the treatment of possible subgroups of certain orders in the interval 144–1728, use is made of the papers† by HÖLDER, COLE. LING and MILLER, in which is determined the simplicity or compositeness of all groups of orders < 2000.

Chiefly in the duplicate proofs, use is made of the lists of all transitive substitution-groups of a given degree, with the following reference numbers:

COLE, 1, Bulletin of the American Mathematical Society, ser. 1, vol. 2, (1903), pp. 250–8.

COLE, 2. Quarterly Journal of Mathematics, vol. 27 (1895), pp. 39–50.

MILLER, 1, Quarterly Journal, vol. 28 (1896), pp. 193–231.

MILLER, 2, Proceedings of the London Mathematical Society, vol. 28 (1897), pp. 533–544.

* DICKSON, Proceedings of the London Mathematical Society, vol. 31 (1899), pp. 30–68; vol. 32 (1900), pp. 3–10.

† For references, see American Journal, vol. 22 (1900), p. 13. It may be noted that the orders $792 = 2^3 \cdot 3^2 \cdot 11$ and $1008 = 2^4 \cdot 3^2 \cdot 7$ were overlooked by BURNSIDE.

MILLER, 3, American Journal of Mathematics, vol. 20 (1898), pp. 229–241.

KUHN, Manuscript list of the imprimitive groups of degree 15.

The following table (referred to as "the table") gives the 114 types of non-conjugate subgroups, other than itself and identity, of G_{25920}, a page reference to their definition, the largest subgroup of G_{25920} in which a given subgroup is self-conjugate, and the number of conjugates within G to a given type.

Group.	Def.	Self-conj. only under.	Numb. conjs.	Group.	Def.	Self-conj. only under.	Numb. conjs.
G_2	III_5	G_{576}	45	K_9^*	II_{382}	H_{108}	240
G_2'	III_6	H_{96}	270	K_9^{**}	II_{383}	H_{216}	120
C_3	130	G_{648}	40	G_{10}	140	G_{20}	1296
C_3'	130	H_{216}	120	C_{12}	134	C_{24}	1080
C_3''	130	H_{108}	240	K_{12}	135	G_{24}^*	1080
C_4^3	III_{18}	G_{96}	270	D_{12}'	135	K_{36}^*	720
C_4^5	III_{18}	G_{16}'	1620	D_{12}^*	135	G_{24}^*	1080
G_4^3	III_4	G_{64}	405	G_{12}	136	G_{24}^*	1080
K_4'	III_6	H_{96}	270	G_{12}	136	G_{48}	540
K_4'''	III_6	H_{96}	270	G_{12}'	136	H_{96}	270
K_4^*	III_{15}	G_{48}	540	G_{16}	III_3	G_{960}	27
G_5	139	G_{20}	1296	G_{16}''	III_7	J_{32}^3	810
C_6	133	G_{72}	360	H_{16}'	III_7	J_{32}^3	810
C_6'	133	K_{36}^*	720	H_{16}^3	III_5	G_{64}	405
C_6'''	133	K_{36}^*	720	J_{16}^3	III_5	G_{64}	405
C_6^*	133	G_{24}^*	1080	F_{16}	III_{10}	G_{96}	270
D_6	133	G_{36}	720	K_{18}	137	K_{54}	480
D_6'	133	H_{108}	240	K_{18}^*	137	K_{36}^*	720
D_6''	133	K_{36}^*	720	K_{18}^{**}	137	H_{108}	240
G_8	III_5	G_{192}	135	G_{18}^*	137	H_{108}	240
G_8''	III_7	H_{96}	270	H_{18}^*	137	K_{36}^*	720
G_8^3	III_4	H_{192}	135	H_{18}^{**}	137	H_{216}	120
H_8^3	III_4	G_{64}	405	G_{18}^{**}	137	G_{36}^{**}	720
K_8	III_6	J_{32}^3	810	G_{20}	140	G_{20}	1296
J_8	III_{12}	G_{32}	810	C_{24}	135	G_{72}	360
F_8''''	III_{13}	G_{288}	90	G_{24}^*	133	G_{24}^*	1080
L_8	III_{15}	G_{16}'	1620	G_{24}^3	141	G_{48}	540
T_8	III_{15}	G_{16}'	1620	L_{24}	141	G_{48}	540
C_9	II_{385}	H_{27}	960	T_{24}	141	G_{48}	540
K_9	II_{382}	G_{162}	160	G_{24}''	142	H_{96}	270

Group.	Def.	Self-conj. only under.	Numb. conjs.	Group.	Def.	Self-conj. only under.	Numb. conjs.
G_{24}	142	H_{48}	540	G^{**}_{72}	148	G^{**}_{72}	360
L^{*}_{24}	142	G''_{48}	540	G_{80}	157	G_{160}	162
F_{24}	144	G'_{72}	360	G_{81}	II_{372}	G_{162}	160
F'_{24}	144	G'_{72}	360	G_{96}	150	G_{288}	90
F^{*}_{24}	144	G_{288}	90	J_{96}	157	G_{576}	45
G_{27}	II_{377}	G_{648}	40	L_{96}	157	G_{576}	45
H_{27}	II_{377}	G_{81}	320	H_{96}	136	H_{96}	270
K_{27}	II_{377}	H_{648}	40	G_{108}	158	G_{216}	120
G_{32}	III_{5}	G_{576}	45	H_{108}	130	H_{648}	40
J^{3}_{32}	III_{5}	G_{64}	405	K'_{108}	158	H_{216}	120
H^{3}_{32}	III_{5}	G_{64}	405	K''_{108}	158	H_{216}	120
K^{*}_{36}	133	K^{*}_{36}	720	G_{120}	153	G_{120}	216
K^{**}_{36}	147	H_{216}	120	G'_{120}	153	G'_{120}	216
G^{**}_{36}	138	G^{**}_{72}	360	G_{160}	157	G_{160}	162
H^{**}_{36}	147	G^{**}_{72}	360	G_{162}	II_{373}	G_{162}	160
G_{48}	136	G_{48}	540	G_{192}	III_{21}	G_{192}	135
G''_{48}	143	H_{96}	270	H_{192}	161	G_{576}	45
H_{48}	143	H_{96}	270	G_{216}	159	G_{648}	40
H''_{48}	149	H_{96}	270	H_{216}	131	H_{216}	120
F_{48}	150	G_{96}	270	G_{288}	147	G_{576}	45
G_{54}	151	G_{648}	40	H_{324}	163	H_{648}	40
K_{54}	138	H_{108}	240	G_{360}	163	G_{720}	36
K'_{54}	151	H_{216}	120	G_{576}	III_{21}	G_{576}	45
G_{60}	III_{3}	G_{120}	216	G_{648}	II_{372}	G_{648}	40
G'_{60}	153	G'_{120}	216	H_{648}	II_{380}	H_{648}	40
G_{64}	III_{3}	G_{192}	135	G_{720}	163	G_{720}	36
G_{72}	133	G_{72}	360	G_{960}	III_{3}	G_{960}	27

Possible orders of subgroups of G.

1. The number of divisors of $25920 = 2^6 \cdot 3^4 \cdot 5$ is $(6+1)(4+1)(1+1) = 70$. These divisors (of which 30 are < 100) are

1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 27, 30, 32, 36, 40, 45, 48, 54, 60, 64, 72, 80, 81, 90, 96, 108, 120, 135, 144, 160, 162, 180, 192, 216, 240, 270, 288, 320, 324, 360, 405, 432, 480, 540, 576, 648, 720, 810, 864, 960, 1080, 1296, 1440, 1620, 1728, 2160, 2592, 2880, 3240, 4320, 5184, 6480, 8640, 12960, 25920.

Orders immediately excluded.

2. THEOREM. *The group G contains no subgroup of one of the following* 16 *orders:* 15, 30, 40, 45, 90, 135, 240, 270, 405, 540, 1440, 3240, 4320, 5184, 6480, 8640, 12960.

By SYLOW's theorem, a group of order 15, 40, 45 or 135 contains a single (and hence self-conjugate) subgroup of order 5. But, within G, a Γ_5 is self-conjugate only under a subgroup of Γ_{20} by I_{138}. Likewise, a Γ_{90} contains 1 or 6 conjugate Γ_5; Γ_{270} contains 1 or 6 conjugate Γ_5; Γ_{240} contains 1, 6, or 16; Γ_{540} contains 1, 6, or 36; Γ_{1440} contains 1, 6, 16, 36, or 96; but in no case is the quotient of the order of the group by the number of conjugates a divisor of 20.

By SYLOW's theorem, a Γ_{30} contains 1 or 6 conjugate Γ_5 and 1 or 10 conjugate Γ_3. But any Γ_5 is self-conjugate in at most a Γ_{20}; while a Γ_3 is self-conjugate only under a subgroup of order 648, 216, or 108, (by I_{138} or § 5 below). Hence a subgroup Γ_{30} must contain 6 conjugate Γ_5 and 10 conjugate Γ_3 and hence at least $1 + 24 + 20$ operators.*

A Γ_{405} contains a single Γ_{81} by SYLOW's theorem, whereas within G any Γ_{81} is self-conjugate only under a group of order 162 by II_{373}.

The final six orders for subgroups are excluded since their indices under G are $\leqq 8$, while 25920 does not divide 8!.

The subgroups of order 3.

3. THEOREM. *Within G, the cyclic subgroups of order* 3 *fall into* 3 *distinct sets of conjugate subgroups, representatives of which are*

$$(1) \qquad C_3 = (L_{1,1}), \qquad C_3' = (L_{1,1}L_{2,1}), \qquad C_3'' = (L_{1,-1}L_{2,1}).$$

They are self-conjugate only under G_{648}, H_{216} and H_{108}, respectively.

The theorem follows from I_{138} except as to the characterization of the groups of order 648, 216, 108. The operators of G commutative with $L_{1,1}$ form G_{648} of II_{372}, while $L_{1,1}$ is not conjugate with its inverse by I_{138}. In view of I_{115}, the only homogeneous abelian substitutions commutative with $L_{1,-1}L_{2,1}$ are $U = [k, 0, c, \gamma]T_{2,\,\pm 1}$. Those transforming $L_{1,-1}L_{2,1}$ into its inverse are $V = UP_{12}$. Hence

$$(2) \qquad H_{108} = \{K_{27}, T_{2,-1}, P_{12}\} \equiv \{[k, 0, c, \gamma], T_{2,-1}, P_{12}\}.$$

Also, by I_{115}, those commutative with $L_{1,1}L_{2,1}$ are the U and V; while those transforming $L_{1,1}L_{2,1}$ into its inverse are seen to be

* Otherwise excluded since Γ_{30} contains a cyclic C_{15}, HÖLDER, Mathematische Annalen, Bd. 43 (1893), p. 412.

$$(3) \qquad \pm \begin{Vmatrix} 1 & \gamma_{11} & -\alpha_{21}\alpha_{22} & \gamma_{12} \\ 0 & -1 & 0 & \alpha_{21}\alpha_{22} \\ \alpha_{21} & \gamma_{21} & \alpha_{22} & \gamma_{22} \\ 0 & -\alpha_{12} & 0 & -\alpha_{22} \end{Vmatrix} \quad \begin{matrix} \alpha_{21}^2 \equiv \alpha_2^2 \equiv 1, \\ \gamma_{21} \equiv \gamma_{11}\alpha_{21} + \gamma_{12}\alpha_{22} + \gamma_{22}\alpha_{21}\alpha_{22}. \end{matrix}$$

Hence the 108 operators (3) together with those of H_{108} form * H_{216}.

Conjugacies among the operators of H_{108}, and among those of H_{216}.

4. By Π_{372}, $U^2 = [-k, 0, c \pm c, -\gamma]$, whence $U^6 = I$. Now $U^2 = U^{-1}$ if and only if the upper signs hold, namely, $U = [k, 0, c, \gamma]$. Hence of the operators U, $[k, 0, c, \gamma]$ is of period 3 if not the identity $[0, 0, 0, 0]$; $[0, 0, c, 0]\,T_{2,-1}$ is of period 2; $[k, 0, c, \gamma]\,T_{2,-1}$ is of period 6 if k and γ are not both $\equiv 0$. The operators $[k, 0, c, \gamma]$ are all commutative by Π_{377}. Now I, P_{12}, $T_{2,-1}$ and $P_{12}T_{2,-1}$ transform $[k, 0, c, \gamma]$ into respectively

$$[k, 0, c, \gamma], \qquad [\gamma, 0, c, k], \qquad [k, 0, -c, \gamma], \qquad [\gamma, 0, -c, k].$$

Of these four, those which are distinct form a complete set of conjugates under H_{108}. Next, the operators of period 2 are all conjugate with $T_{2,-1}$, since $T_{2,-1}$ and $[0, 0, -1, 0]$ transform $[0, 0, 1, 0]\,T_{2,-1}$ into $[0, 0, -1, 0]\,T_{2,-1}$ and $T_{2,-1}$ respectively. Consider finally the operators U of period 6. Applying (12) of Π_{377}, we find that $[0, 0, -c, 0]$ transforms $[k, 0, c, \gamma]\,T_{2,-1}$ into $[k, 0, 0, \gamma]\,T_{2,-1}$. The latter is transformed into $[\gamma, 0, 0, k]$ by P_{12}.

For $V = [k, 0, c, \gamma]\,T_{2,\pm 1}P_{12}$, we have

$$V^2 = [k + \gamma, 0, c \pm c, k + \lambda].$$

Hence V is of period 2 or 6. Those of period 2 are $[-\gamma, 0, 0, \gamma]\,P_{12}$ and $[-\gamma, 0, c, \gamma]\,T_{2,-1}P_{12}$. The former is transformed into P_{12} by $[0, 0, 0, \gamma]$, the latter into $[0, 0, c, 0]\,T_{2,-1}P_{12} \equiv S_c$ by $[0, 0, 0, \gamma]$. But $[0, 0, -c, 0]$ transforms S_c into $S_0 = P_{12}T_{2,-1}$. The operators V of period 6 are $[k, 0, c, \gamma]\,T_{2,-1}P_{12}$, $k + \gamma \neq 0$, and $[k, 0, c, \gamma]\,P_{12}$, $k + \gamma$ and c not both 0. The first is transformed into $[k + \gamma, 0, 0, 0]\,T_{2,-1}P_{12}$ by $[0, 0, -c, \gamma]$; the second into $[k + \gamma, 0, c, 0]\,P_{12}$ by $[0, 0, 0, \gamma]$.

THEOREM.† *The operators of H_{108} are of period 1, 2, 3, or 6. Those of period 1 or 3 are $[k, 0, c, \gamma]$ and are commutative. Those of period 2*

* By Π_{383}, H_{216} is the largest subgroup transforming K_9^{**} into itself.

† For an ultimate classification, not needed here, we note that the operators of period 3 fall into the following distinct sets of conjugates:

$$\{[\gamma, 0, 0, \gamma]\}, \quad \{[\gamma, 0, 1, \gamma], [\gamma, 0, -1, \gamma]\}, \quad \{[k, 0, 0, \gamma], [\gamma, 0, 0, k]\}, \quad (k \neq \gamma).$$
$$\{[k, 0, 1, \gamma], [k, 0, -1, \gamma], [\gamma, 0, 1, k], [\gamma, 0, -1, k]\}$$

The operators of period 6 are conjugate with $[\pm 1, 0, 0, 0]T_{2,-1}$, $[\pm 1, 0, 0, 1]\,T_{2,-1}$, $[-1, 0, 0, -1]\,T_{2,-1}$, $[\pm 1, 0, 0, 0]\,P_{12}$, or $[k, 0, 1, 0]\,P_{12}$, no two of which are conjugate.

are conjugate within H_{108} *with* $T_{2,-1}$, P_{12}, *or* $P_{12}T_{2,-1}$. *Those of period* 6 *are conjugate within* H_{108} *with*

$$[k,0,0,\gamma]T_{2,-1},\ [\pm 1,0,0,0]T_{2,-1}P_{12},\ [k,0,\gamma,0]P_{12}\quad (k,\gamma \text{ not both} \equiv 0).$$

5. Operator (3) is transformed by $T_{2,-1}$ into a similar operator with α_{21} replaced by $-\alpha_{21}$. We therefore take $\alpha_{21} \equiv +1$. Let first $\alpha_{22} \equiv -1$. The resulting operator (3) is transformed by $L_{1,-\gamma_{11}}L_{2,\gamma_{22}}$ into

$$W_\gamma = \pm \begin{bmatrix} 1 & 0 & 1 & \gamma \\ 0 & -1 & 0 & -1 \\ 1 & -\gamma & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \qquad (\gamma \equiv \gamma_{21} - \gamma_{12}).$$

It is of period 2 if and only if $\gamma \equiv 0$. Now $P_{12}T_{2,-1}$ transforms W_1 into W_{-1}; while $W_1^2 = [-1,0,1,1]$ is of period 3 and differs from W_1^{-1}. We obtain therefore two reduced forms: W_0 of period 2 and W_1 of period 6. Let next $\alpha_{22} \equiv +1$. The resulting operator (3) is transformed by $L_{1,-\gamma_{11}}L_{2,-\gamma_{22}}$ into

$$Z_\delta \equiv \pm \begin{bmatrix} 1 & 0 & -1 & \delta \\ 0 & -1 & 0 & 1 \\ 1 & \delta & 1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix} \qquad (\delta \equiv -\gamma_{12} - \gamma_{21}).$$

The latter is transformed into $Z_0 \equiv T_{2,-1}W_0$ by the operator $[0,0,-\delta,0]$ of H_{216}. Now $Z_0^2 = P_{12}T_{2,-1}$, since W_0 transforms $T_{2,-1}$ into P_{12}. Hence Z_0 is of period 4 in the quotient-group G.

Consider finally the conjugacy of the operators of H_{108} under the group H_{216}. Now W_0 transforms P_{12} into $T_{2,-1}$. Again, W_0 transforms $[k,0,c,\gamma]$ into $[k+\gamma-c,0,k-\gamma,k+\gamma+c]$. Hence W_0 transforms $[k,0,0,\gamma]T_{2,-1}$ into $[k+\gamma,\ 0,\ k-\gamma,\ k+\gamma]P_{12}$. The latter is transformed into $[-k-\gamma,0,k-\gamma,0]P_{12}$ by $[0,0,0,k+\gamma]$. Now $-k-\gamma$ and $k-\gamma$ are not both 0 if k and γ are not.

THEOREM. *The operators of* H_{216} *are of period* 1, 2, 3, 4 *or* 6. *Those of period* 1 *or* 3 *are* $[k,0,c,\gamma]$. *Within* H_{216}, *those of period* 2 *are conjugate with* $P_{12}T_{2,-1}$, $T_{2,-1}$ *or* W_0; *those of period* 4 *are conjugate with* $Z_0 \equiv T_{2,-1}W_0$; *those of period* 6 *are conjugate with* W_1, $[k,0,0,\gamma]T_{2,-1}$, *or* $[\pm 1,0,0,0]T_{2,-1}P_{12}$, *where* k *and* γ *are not both* 0.

The subgroups of order 6.

6. THEOREM. *Within* G, *the cyclic subgroups of order* 6 *fall into* 4 *distinct sets of conjugate subgroups, representations of which are*

$$(4)\qquad \begin{aligned} C_6 &= (L_{1,1}T_{1,-1}), & C_6' &= (L_{1,1}L_{2,-1}T_{1,-1}),\\ C_6'' &= (L_{1,1}L_{2,1}T_{1,-1}), & C_6^* &= (P_{12}L_{1,-1}T_{1,-1}). \end{aligned}$$

They are self-conjugate only under G_{72}, K_{36}^*, K_{36}^* *and* G_{24}^*, *respectively:*

$$(5)\qquad G_{72} = \left\{ \pm \begin{bmatrix} 1 & k & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & \gamma \\ 0 & 0 & \beta & \delta \end{bmatrix}, \quad (\alpha\delta - \beta\gamma \equiv 1) \right\},$$

$$(6)\qquad K_{36}^* = (K_9^*, T_{2,-1}, P_{12}) = \{[k, 0, 0, \gamma]R, (R = I, T_{2,-1}, P_{1,2}, T_{2,-1}P_{12})\},$$

$$(7)\qquad G_{24}^* = \{C, CT_{2,-1}, CD, CT_{2,-1}D, (C \text{ ranging over } C_6^*)\},$$

$$(7_a)\qquad D = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & -1 & 0 & -1 \\ 1 & 1 & -1 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}.$$

The theorem follows from I_{138} except as to the characterization of the groups of order 72, 36, 24. By I_{116}, $L_{1,1}T_{1,-1}$ is commutative only with the substitutions of G_{72}. By I_{115}, $L_{1,1}T_{1,-1}$ is not conjugate with its inverse. By I_{116}, $L_{1,1}L_{2,-1}T_{1,-1}$ is commutative only with $[\gamma, 0, 0, \gamma_{22}]T_{2,\pm 1}$; it is transformed into its inverse by P_{12}. Also $L_{1,1}L_{2,1}T_{1,-1}$ is commutative only with $[\gamma, 0, 0, \gamma_{22}]T_{2,\pm 1}$, while it is transformed into $L_{1,1}L_{2,1}T_{2,-1}$, the same as the former in the quotient-group G, by P_{12}. Finally, $P = P_{12}L_{1,-1}T_{1,-1}$ is commutative only with its powers and their products by $T_{2,-1}$, which transforms P into $PT_{1,-1}T_{2,-1}$. The only homogeneous substitutions transforming P into P^{-1} are found to be the 6 operators

$$(8)\qquad \pm \begin{bmatrix} 1 & \gamma & \pm 1 & \pm 1 \pm \gamma \\ 0 & -1 & 0 & \mp 1 \\ \pm 1 & \pm 1 \pm \gamma & -1 & 1 - \gamma \\ 0 & \mp 1 & 0 & 1 \end{bmatrix},$$

which may be written as the products CD, C ranging over C_6^*.

7. THEOREM. *Within* G, *the non-cyclic groups of order* 6 *fall into* 3 *distinct sets of conjugate subgroups, representatives of which are*

$$(9)\qquad D_6 = (L_{1,1}L_{2,1}, W_0), \; D_6' = (L_{1,-1}L_{2,1}, P_{12}), \; D_6'' = (L_{1,-1}L_{2,1}, P_{12}T_{2,-1}).$$

They are self-conjugate only under G_{36}, H_{108}, K_{36}^*, *respectively, where*

(10) $G_{36} = \{ [k, 0, c, k-c] \Gamma, (\Gamma = I, T_{2,-1} P_{12}, W_0, T_{2,-1} P_{12} W_0) \}$.

The subgroups sought are dihedron $G_{2\cdot 3}$ generated by A and B with

(11) $$A^3 = I, \qquad B^2 = I, \qquad BAB = A^{-1}.$$

We may assume that A is $L_{1,1}$, $L_{1,1} L_{2,1}$ or $L_{1,-1} L_{2,1}$ (§ 3). But $L_{1,1}$ is excluded since it is not conjugate with its inverse within G.

The cyclic group generated by $A = L_{1,-1} L_{2,1}$ is self-conjugate only under H_{108} (§ 3), whose operators of period 2 are conjugate with $T_{2,-1}$, P_{12}, or $P_{12} T_{2,-1}$ (§ 4). The last two transform A into A^{-1}, but $T_{2,-1}$ transforms A into itself and is excluded.

The cyclic group generated by $A = L_{1,1} L_{2,1}$ is self-conjugate only under H_{216} (§ 4), whose operators of period 2 are conjugate with $P_{12} T_{2,-1}$, $T_{2,-1}$, or W_0 (§ 5). The first two transform A into itself and are excluded, while W_0 transforms A into A^{-1}.

An operator which transforms D_6' into itself must transform the subgroup $(L_{1,-1} L_{2,1})$ into itself and hence belong to H_{108}. Moreover, it must transform P_{12} into one of the 3 operators $L_t = L_{1,-t} L_{2,t} P_{12}$ of period 2 in D_6'. Now $U \equiv [k, 0, c, \gamma] T_{2,\,*1}$ transforms P_{12} into $L_{k-\gamma}$; while $V \equiv [k, 0, c, \gamma] T_{2,\,*1} P_{12}$ transforms P_{12} into $L_{\gamma-k}$. Hence H_{108} transforms D_6' into itself.

For D_6'', we seek the operators of H_{108} which transform $P_{12} T_{2,-1}$ into one of the 3 operators $M_t = L_{1,-t} L_{2,t} P_{12} T_{2,-1}$ of period 2 in D_6''. For U the conditions are $c \equiv 0$, $t \equiv k - \gamma$; for V the conditions are $c \equiv 0$, $t \equiv \gamma - k$. Hence D_6'' is self-conjugate only under the group (6).

For D_6, we seek the operators of H_{216} which transform W_0 into one of the 3 operators $N_t \equiv L_{1,t} L_{2,t} W_0$ of period 2 in D_6. For U the conditions are $\pm 1 \equiv +1$, $k \equiv c + \gamma$, $t \equiv -k - \gamma$; for V the conditions are $\pm 1 \equiv -1$, $c \equiv k - \gamma$, $t \equiv -k - \gamma$. For (3) the conditions are

for the upper signs: $\alpha_{22} \equiv -1$, $\alpha_{21} \equiv 1$, $\gamma_{22} \equiv -\gamma_{11}$, $t \equiv \gamma_{12} + \gamma_{21} - \gamma_{11}$.

for the lower signs: $\alpha_{22} \equiv -1$, $\alpha_{21} \equiv -1$, $\gamma_{12} \equiv \gamma_{21}$, $t \equiv \gamma_{22} - \gamma_{21} - \gamma_{11}$;

The resulting operators (3) are respectively

$$[\gamma_{12}, 0, \gamma_{12} - \gamma_{21}, \gamma_{21}] W_0, \qquad [-\gamma_{22}, 0, -\gamma_{22} - \gamma_{11}, \gamma_{11}] T_{2,-1} P_{12} W_0.$$

Hence D_6 is self-conjugate only under the group (10).

The subgroups of order 12.

8. THEOREM. *Within G, every cyclic subgroup of order* 12 *is conjugate with*

(12) $$C_{12} = (M_2 L_{1,1}).$$

The latter is self-conjugate only under the group

$$(13) \qquad C_{24} = (C_{12}, A), \qquad A\colon \xi_2' = \xi_2 + \eta_2,\ \eta_2' = \xi_2 - \eta_2.$$

For proof, see I_{128}. Note that $A^2 = (M_2 L_{1,1})^6 = T_{2,-1}$.

9. THEOREM. *Within G, every non-cyclic commutative subgroup of order* 12 *is conjugate with K_{12}, every dihedral subgroup of order* 12 *is conjugate with D_{12}' or else with D_{12}^*, where*

$$(14) \qquad K_{12} = (C_6^*, T_{2,-1}), \qquad D_{12}' = (C_6', P_{12}), \qquad D_{12}^* = (C_6^*, D).$$

They are self-conjugate only under G_{24}^*, K_{36}^* and G_{24}^*, respectively.

A non-cyclic commutative Γ_{12} or a dihedral Γ_{12} contains a self-conjugate cyclic Γ_6 and an operator of period 2 not in the latter. By § 6 we may take C_6, C_6', C_6'' or C_6^* as the Γ_6.

Since C_6 is self-conjugate only under G_{72}, which contains a single operator $T_{2,-1}$ of period 2, it is to be excluded.

The groups C_6' and C_6'' are each self-conjugate only under K_{36}^*, which contains exactly 7 operators of period 2, viz.,

$$T_{2,-1}, \qquad L_{1,-\gamma} L_{2,\gamma} P_{12}, \qquad L_{1,-\gamma} L_{2,\gamma} T_{2,-1} P_{12}.$$

Since $T_{2,-1}$ and $L_{1,-\gamma} L_{2,\gamma}$ lie in C_6', we may limit the extender to P_{12} and thus obtain D_{12}'. Since $T_{2,-1}$ lies in C_6'', we may limit the extender to $L_{1,-\gamma} L_{2,\gamma} P_{12}$. But $L_{2,\gamma}$ transforms the latter into P_{12} and transforms C_6'' into itself. There results $K_{12}'' = (C_6'', P_{12})$. But $P_{12} L_{2,-1}$ transforms K_{12} into K_{12}'' since it transforms $P_{12} L_{1,-1} T_{1,-1}$ and $T_{2,-1}$ into the generators $P_{12} L_{1,1} L_{2,1} T_{2,-1}$ and $T_{1,-1}$ of (C_6'', P_{12}).

Finally, C_6^* is self-conjugate only under G_{24}^*. The only operators of the latter commutative with $P \equiv P_{12} L_{1,-1} T_{1,-1}$ are those of K_{12}; the remaining operators $P^i D$ and $P^i T_{2,-1} D\,(i = 1, \cdots, 6)$ transform P into P^{-1} (§ 6), the first 6 being of period 2 and the last 6 of period 4.

A dihedral Γ_{12} contains a single cyclic Γ_6. Hence D_{12}^* is self-conjugate only under G_{24}^*. Likewise, a substitution commutative with D_{12}' must belong to K_{36}^*. But all the operators of period 2 in the latter belong to D_{12}'. Hence D_{12}' is self-conjugate under K_{36}^*.

Since K_{12} contains the single self-conjugate cyclic group $(L_{1,1} L_{2,1})$ of order 3, an operator transforming K_{12} into itself must belong to H_{216} (§ 3). Since K_{12} contains the single self-conjugate group

$$(15) \qquad \{I, \quad T_{2,-1}, \qquad P_{12} L_{1,1} L_{2,-1} T_{1,-1}, \qquad P_{12} L_{1,1} L_{2,-1}\}$$

of order 4, an operator transforming K_{12} into itself must belong to a group of order 96 or 64 by III, or by the table. Hence K_{12} is transformed into itself by at most 24 operators. But K_{12} is a subgroup of G_{24}^* and hence self-conjugate under it.

10. THEOREM. *Every subgroup simply isomorphic with the group generated by the two operators R and S subject to the generational relations*

$$R^6 = I, \qquad S^2 = R^3, \qquad S^{-1}RS = R^{-1} \tag{16}$$

is conjugate within G with the group

$$G_{12}^* = \{P^i,\ P^i T_{2,-1} D,\ (P \equiv P_{12} L_{1,-1} T_{1,-1},\ i = 1, \cdots, 6)\}. \tag{17}$$

Within G, G_{12}^* is self-conjugate only under G_{24}^*.

The self-conjugate Γ_6 may be taken to be C_6, C_6', C_6'', or C_6^*. Now C_6' and C_6'' are self-conjugate only under K_{36}^*, which contains no operator of period 4, and hence are excluded.

The group C_6 is self-conjugate only under G_{72}. But an operator (5) is of period 4 if and only if $k \equiv 0$, $\alpha + \delta \equiv 0$ (IV), when it becomes

$$\xi_1' = \xi_1,\ \eta_1' = \eta_1,\ \xi_2' = \alpha\xi_2 + \gamma\eta_2,\ \eta_2' = \beta\xi_2 - \alpha\eta_2 \quad (-\alpha^2 - \beta\gamma \equiv 1). \tag{18}$$

The Γ_{12} must contain 6 operators of period 4 and hence contain every (18). But $L_{1,\lambda} T_{2,\pm 1}$ (18) is neither in C_6 nor of the form (18) if $\gamma \neq 0$, $\beta \neq 0$. Hence also C_6 is to be excluded.

Finally, C_6^* is self-conjugate only under G_{24}^*. Now $P^i T_{2,-1} D$ and no further operators of G_{24}^* are of period 4, where $P \equiv P_{12} L_{1,-1} T_{1,-1}$, since

$$(P^i T_{2,-1} D)^2 = P_{12} L_{1,1} T_{2,-1} L_{2,-1} = P^3.$$

By (3), 6, $P^i T_{2,-1} D$ transforms P into P^{-1}. Hence G_{12}^* satisfies the conditions.

11. THEOREM. *Within O, every subgroup simply isomorphic with the alternating group on 4 letters is conjugate with one of the two groups*

$$\begin{aligned} G_{12} = \{&I,\ C_1C_3,\ B_3,\ B_3C_1C_3,\ C_2C_4B_iW,\ C_1C_2C_3C_4B_iW,\\ &B_iC_2C_3W^2,\ B_iC_3C_4W^2,\ (i = 2, 4)\}, \end{aligned} \tag{19}$$

$$G_{12}' = \{\Gamma,\ \Gamma(\xi_2\xi_4\xi_5),\ \Gamma(\xi_2\xi_5\xi_4),\ (\Gamma = I,\ C_2C_4,\ C_2C_5,\ C_4C_5)\}. \tag{20}$$

They are self-conjugate only under the respective groups

$$G_{48} = \{\Gamma,\ \Gamma B_2 W,\ \Gamma W^2 B_2 \equiv \Gamma B_2 C_1 C_2 W^2,\ (\Gamma \textit{ ranging over } G_{16}')\}, \tag{21}$$

$$\begin{aligned} H_{96} = \{&\Gamma,\ B_3\Gamma,\ (\xi_1\xi_3)(\xi_4\xi_5)\Gamma,\ (\xi_1\xi_3)(\xi_2\xi_5)\Gamma,\ (\xi_2\xi_4\xi_5)\Gamma,\ (\xi_2\xi_5\xi_4)\Gamma,\\ &(\Gamma \textit{ ranging over } G_{16})\}. \end{aligned} \tag{22}$$

For the self-conjugate *four-group*, we may take G_4^2, K_4', K_4''' or K_4^*. Within O, these are self-conjugate only under G_{64}, H_{96}, H_{96}, G_{48}, respectively. Hence G_4^2 is excluded. The only operators of period 3 in H_{96} are the last 8 operators (20). They must therefore all occur in the group of order 12. They extend K_4' to G_{12}' and K_4''' to H_{48}, defined by (43), so that K_4''' is excluded. The only operators of period 3 in G_{48} are seen (see § 22) to be the last 8 operators (19). Combined with K_4^*, they give G_{12}.

12. *Summary of the subgroups of order* 12. All have now been determined since the five types * of groups of order 12 were examined in §§ 8–11.

THEOREM. *Every existing type of group of order* 12 *is represented among the subgroups of* G. *Within* G, *they fall into seven distinct sets of conjugates, two of the dihedron type, two of the type of the alternating group on four letters, and one of each of the three remaining types.*

The subgroups of order 18.

13. THEOREM. *The group* G *contains exactly seven distinct sets of conjugate subgroups of order* 18, *representatives of which are*

$$
\begin{aligned}
&K_{18} = (K_9,\ T_{2,-1}), \qquad K_{18}^{*} = (K_9^{*},\ T_{2,-1}), \qquad K_{18}^{**} = (K_9^{**},\ T_{1,-1}),\\
(23)\quad &G_{18}^{*} = (K_9^{*},\ P_{12}), \qquad H_{18}^{*} = (K_9^{*},\ P_{12}T_{2,-1}),\\
&H_{18}^{**} = (K_9^{**},\ P_{12}T_{2,-1}), \qquad G_{18}^{**} = (K_9^{**},\ W_0).
\end{aligned}
$$

A Γ_{18} contains a single (self-conjugate) subgroup Γ_9. But within G a cyclic Γ_9 is self-conjugate only under a Γ_{27} by Π_{385}. As the Γ_9 we may therefore take one of the non-cyclic groups K_9, K_9^{*}, K_9^{**} (Π_{386}).

The group K_9 of the operators $[k, 0, c, 0]$ is self-conjugate only under G_{162} by Π_{383}, which contains exactly the 9 operators $[0, \alpha, \gamma, 0]\,T_{2,-1}$ of period 2. If $\alpha = 0$, the latter is transformed into $T_{2,-1}$ by $[0, 0, -\gamma, 0]$; if $\alpha \neq 0$ it is transformed by the substitution $L_{2,\gamma/\alpha}$ of G_{162} into $[0, \alpha, 0, 0]\,T_{2,-1}$. The latter is transformed into $T_{2,-1}$ by the following substitution of G_{162}:

$$(24)\qquad \xi_1' = \xi_1 - \alpha\xi_2, \qquad \eta_2' = \eta_2 + \alpha\eta_1.$$

The group K_9^{*} of the operators $[k, 0, 0, \gamma]$ is self-conjugate only under H_{108} by Π_{384}. By § 4 the operators of period 2 of H_{108} are conjugate within H_{108} with P_{12}, $P_{12}T_{2,-1}$, or $T_{2,-1}$. Of the resulting groups G_{18}^{*}, H_{18}^{*}, K_{18}^{*}, the first two each contain exactly 3 substitutions of period 2 and the third only one. The 3 of the first and the 3 of the second have the characteristic determinants $(\rho^2 - 1)^2$ and $(\rho^2 + 1)^2$, respectively. Hence no two of these three groups are conjugate under linear transformation.

The group K_9^{**} of the operators $[-\gamma, 0, c, \gamma]$ is self-conjugate only under H_{216} (see foot-note to § 3). The operators of period 2 of H_{216} are conjugate with $P_{12}T_{2,-1}$, $T_{2,-1}$ or W_0 (§ 5). No two of the resulting groups $(K_9^{**},\ Q)$, $Q = P_{12}T_{2,-1}$, $T_{2,-1}$ or W_0, are conjugate within G. Indeed, $[-\gamma, 0, c, \gamma]\,W_0$ is of period 2 if and only if $c + \gamma \equiv 0$, so that $(K_9^{**},\ W_0)$ contains exactly 3

*CAYLEY, American Journal of Mathematics, vol. 11 (1889), pp. 151–3. In his substitution V of $B4$, (dj) should read (gj). In MILLER's list, Quarterly Journal, vol. 28 (1896), p. 255, the group 12_1 should have $(ag)(bh)(ci)(dj)(ek)(fl)$ as the second generator.

operators of period 2, each with the characteristic determinant $(\rho^2+1)^2$. Since $[-\gamma, 0, c, \gamma]\,T_{2,-1}$ is of period 2 if and only if $\gamma \equiv 0$, the group $(K_9'', T_{2,-1})$ contains exactly 3 operators of period 2, each with the characteristic determinant $(\rho^2-1)^2$. Finally $[-\gamma, 0, c, \gamma]\,P_{12}\,T_{2,-1}$ is of period 2 for every c and γ, so that $(K_9'', P_{12}\,T_{2,-1})$ contains exactly 9 operators of period 2.

14. THEOREM. *Within G, the group K_{18} is self-conjugate only under*

$$(25)\quad K_{54} = (K_{27}, T_{2,-1}) = \{[k, 0, c, \gamma]\,T_{2,\pm 1} \quad (k, c, \gamma = 0, 1, 2)\}.$$

The operators of K_{18} not in K_9 are $[k, 0, c, 0]\,T_{2,-1}$. The latter is of period 2 if and only if $k \equiv 0$. The general operator (6) of Π_{373} in G_{162} transforms $T_{2,-1}$ into an operator $[0, 0, c, 0]\,T_{2,-1}$ if and only if $\alpha_{12} \equiv 0$, $\gamma_{12} \equiv c\alpha_{22}$, provided we set $\alpha_{11} = +1$, as we may. The operators (6) with $\alpha_{12} \equiv 0$ form K_{54}.

15. THEOREM. *Within G, the group K_{18}^* is self-conjugate only under K_{36}^*.*

The only operator of period 2 in K_{18}^* is $T_{2,-1}$. It thus remains to determine the operators of H_{108} which are commutative with $T_{2,-1}$. For U of § 3, the condition is $c \equiv 0$. For V the condition is $c \equiv 0$.

16. THEOREM. *Within G, the group G_{18}^* is self-conjugate only under H_{108}.*

The only operators of period 2 in G_{18}^* are $N_t = [-t, 0, 0, t]\,P_{12}$. But U and V of § 3 transform P_{12} into $N_{k-\gamma}$ and $N_{\gamma-k}$, respectively.

17. THEOREM. *Within G, the group H_{18}^* is self-conjugate only under K_{36}^*.*

The only operators of period 2 in H_{18}^* are $Q_t = [-t, 0, 0, t]\,P_{12}\,T_{2,-1}$. Now U transforms $P_{12}\,T_{2,-1}$ into Q_t if and only if $c \equiv 0$, $t \equiv k-\gamma$; while V transforms $P_{12}\,T_{2,-1}$ into Q_t if and only if $c \equiv 0$, $t \equiv \gamma - k$.

18. THEOREM. *Within G, the group K_{18}'' is self-conjugate only under H_{108}.*

The only operators of period 2 in K_{18}'' are $R_t = [0, 0, t, 0]\,T_{2,-1}$. Now U and V transform $T_{2,-1}$ into $R_{\pm c}$, so that K_{18}'' is certainly self-conjugate under H_{108}. But $T_{2,-1}\,Y = Y\,R_t$, where Y is of the form (3), requires $\alpha_{21} \equiv 0$ and is impossible.

19. THEOREM. *Within G, the group H_{18}'' is self-conjugate only under H_{216}.*

The 9 operators $S_{c,\gamma} = [-\gamma, 0, c, \gamma]\,P_{12}T_{2,-1}$ belonging to H_{18}'' but not to K_9'' are all of period 2 (end of § 13). Now U and V transform $P_{12}T_{2,-1}$ into $S_{\pm c,\,k-\gamma}$ and $S_{\pm c,\,\gamma-k}$, respectively; while (3) transforms it into $S_{c,\gamma}$ where

$$c \equiv \alpha_{21}(\pm\gamma_{22} - \gamma_{11}) \mp \gamma_{12} - \gamma_{21}, \qquad \gamma \equiv \alpha_{21}(\mp\gamma_{12} - \gamma_{21}) - (\pm\gamma_{22} - \gamma_{11}).$$

20. THEOREM. *Within G, the group G_{18}'' is self-conjugate only under*

$$(26)\quad \begin{aligned} G_{36}'' &= (G_{18}'', T_{2,-1}P_{12}) \\ &= \{[-\gamma, 0, c, \gamma]\,R,\ (R = I,\ W_0,\ T_{2,-1}P_{12},\ W_0\,T_{2,-1}P_{12})\}, \end{aligned}$$

whose 36 operators may be exhibited in the explicit form

$$\pm\begin{bmatrix} 1 & \gamma_{11} & 0 & \gamma_{12} \\ 0 & 1 & 0 & 0 \\ 0 & \gamma_{12} & 1 & -\gamma_{11} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \pm\begin{bmatrix} 0 & -\gamma_{22} & -1 & \gamma_{12} \\ 0 & 0 & 0 & -1 \\ 1 & \gamma_{12} & 0 & \gamma_{22} \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

(27)

$$\pm\begin{bmatrix} 1 & \gamma_{11} & \mp 1 & \gamma_{12} \\ 0 & -1 & 0 & \pm 1 \\ \mp 1 & -\gamma_{12} & -1 & \gamma_{11} \\ 0 & \pm 1 & 0 & 1 \end{bmatrix}.$$

The only operators of period 2 in $G_{18}^{\cdots}$ are $Y_t = [t, 0, t, -t]\, W_0$. We seek the conditions under which $S^{-1} W_0 S = Y_t$, where S belongs to H_{216}. For $S = U$, the upper signs must be taken and $\gamma \equiv -k$, $t \equiv -k - c$. For $S = V$, the lower signs must be taken and $\gamma \equiv -k$, $t \equiv c - \gamma$. For (3), S must be of the third type and $t \equiv -\gamma_{12}$ or γ_{11} according as the upper or the lower signs hold.

The subgroups of orders 10 *and* 20.

21. A group of order 10 or 20 contains a single (self-conjugate) subgroup of order 5. Within G the groups of order 5 are all conjugate, and each is self-conjugate only under a Γ_{20} by I_{138} or as shown below.

Consider the group G_5 generated by K of I_{137}. The homogeneous substitution K is of period 10 since $K^5 = T_{1,-1} T_{2,-1}$. The corresponding operator in the quotient-group G is of period 5. Within G, K, K^2, K^3 and K^4 are all conjugate, there being a single set of conjugate operators of period 5 by I_{138}. The conditions for $KS = SK^2 T_{1,-1} T_{2,-1}$ in the homogeneous group are seen to require that

$$S = \pm\begin{bmatrix} \alpha & \gamma & \beta+\gamma+\delta & -\alpha+\gamma-\delta \\ \beta & \delta & -\alpha+\beta-\delta & \alpha+\beta+\gamma \\ \alpha+\beta+\gamma & -\alpha+\beta-\delta & -\delta & -\beta \\ -\alpha+\gamma-\delta & \beta+\gamma+\delta & -\gamma & -\alpha \end{bmatrix}.$$

The abelian conditions on S then reduce to

$$\beta^2 + \gamma^2 - \alpha^2 - \delta^2 - \alpha\beta - \alpha\gamma - \beta\delta - \gamma\delta \equiv 1,$$

$$-\beta^2 - \gamma^2 - \alpha^2 - \delta^2 + \alpha\beta - \alpha\gamma - \beta\delta + \gamma\delta + \alpha\delta + \beta\gamma \equiv 0.$$

If $\alpha \equiv 0$, we find by addition that $\delta^2 + \beta\delta + \beta\gamma \equiv 1$. Since $\delta \equiv 0$ is excluded, we may set $\delta \equiv +1$, placing the ambiguity in sign before the matrix. From

$$\beta(1+\gamma) \equiv 0, \qquad \beta^2 + \gamma^2 - \beta - \gamma \equiv -1,$$

it follows that $\gamma \neq 1$ or 0. With $\gamma \equiv -1$, $\beta \equiv 0$ or 1, giving respectively

$$S_1 = \pm \begin{Vmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & -1 \\ -1 & -1 & -1 & 0 \\ 1 & 0 & 1 & 0 \end{Vmatrix}, \qquad S_2 = \pm \begin{Vmatrix} 0 & -1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 1 & 1 & 1 & 0 \end{Vmatrix}.$$

If $\alpha \equiv 1$, then $(\beta, \gamma, \delta) = (-1, 0, 0), (-1, 1, 0)$, or $(0, 0, -1)$, giving respectively

$$S_3 = \pm \begin{Vmatrix} 1 & 0 & -1 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & -1 & 0 & -1 \end{Vmatrix}, \qquad S_4 = \pm \begin{Vmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{Vmatrix},$$

$$S_5 = \pm \begin{Vmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{Vmatrix}.$$

The five S_i are of period 4 and no one equals the inverse of another, their squares being distinct. Hence G_5 is self-conjugate only under

$$G_{20} = \{K', S_i^j, (i, t = 1, \cdots; j = 1, 2, 3)\}. \tag{28}$$

It contains a single subgroup of order 10:

$$G_{10} = \{K', S_i^2, (i, t = 1, \cdots, 5)\}. \tag{29}$$

THEOREM.* *A subgroup of order* 10 *or* 20 *is conjugate within* G *with* G_{10} *or* G_{20}, *respectively. The latter are self-conjugate only under* G_{20}.

The subgroups of order 24.

22. The single† type of group of order 24 which does not contain a subgroup of order 12 is considered in § 30. Consider here the Γ_{24} which have a (self-conjugate) subgroup Γ_{12}. The 7 distinct sets of conjugate Γ_{12} in G are represented by C_{12}, K_{12}, D'_{12}, D^*_{12}, G^*_{12}, G_{12}, G'_{12}; they are self-conjugate only under C_{24}, G^*_{24}, K^*_{36}, G^*_{24}, G^*_{24}, G_{48}, H_{96}, respectively (§§ 8–11). Hence D'_{12} is excluded, while each of the groups C_{12}, K_{12}, D^*_{12}, and G^*_{12} leads to a single Γ_{24}.

* Compare the corresponding investigation on the orthogonal form O, § 46.

† MILLER, Quarterly Journal, vol. 28 (1896), p. 274.

We next determine the Γ_{24} which contain G_{12}. The 8 operators of period 3 in G_{48} are the last 8 operators (19). Their products by $C_1C_2C_3C_4$ evidently give the 8 operators of period 6, viz.,

$$(30)\qquad B_iW,\quad B_iC_2C_4W,\quad B_iC_1C_4W^2,\quad B_iC_1C_2W^2\quad (i=2,4).$$

The 19 operators of period 2 are

$$(31)\quad \begin{cases} C_1C_3,\ B_3,\ B_3C_1C_3,\ B_3C_2C_4,\ B_3C_1C_2C_3C_4,\ C_2C_4,\ C_1C_2C_3C_4,\ C_1C_0, \\ \qquad\qquad C_3C_0,\ C_1C_5,\ C_3C_5 \\ C_1C_5B_iW,\ C_1C_0B_iW,\ C_iC_5B_iW^2,\ C_iC_0B_iW^2 \end{cases}\quad (i=2,4).$$

The 12 operators of period 4 are

$$(32)\quad \begin{matrix} B_3C_iC_5,\quad B_3C_iC_0,\quad B_jC_3C_5W^2,\quad B_jC_3C_0W^2, \\ B_2C_4C_5W,\quad B_2C_4C_0W,\quad B_4C_2C_5W,\quad B_4C_2C_0W \end{matrix}\quad (i=1,3;\ j=2,4).$$

Now C_2C_4, C_1C_5 and C_1C_0 extend G_{12} to the respective groups

$$(33)\quad G_{24}^3=\left\{\begin{matrix}\Gamma,\ C_1C_3\Gamma,\ C_2C_4\Gamma,\ C_1C_2C_3C_4\Gamma\quad(\Gamma=I,\ B_3,\ B_2W,\ B_4W)\\ B_iC_1C_2W^2,\ B_iC_2C_3W^2,\ B_iC_1C_4W^2,\ B_iC_3C_4W^2\quad(i=2,4)\end{matrix}\right\},$$

$$(34)\quad L_{24}=\left\{\begin{matrix}\Gamma,\ C_1C_3\Gamma,\ C_1C_5\Gamma,\ C_3C_5\Gamma,\ B_iC_2C_3W^2,\ B_iC_3C_4W^2,\\ C_1C_0B_iW,\ C_3C_0B_iW,\ C_2C_4B_iW,\ C_1C_2C_3C_4B_iW,\\ C_iC_0B_iW^2,\ C_2C_5B_4W^2,\ C_4C_5B_2W^2\end{matrix}\right\},$$

$$(35)\quad T_{24}=\left\{\begin{matrix}\Gamma,\ C_1C_3\Gamma,\ C_1C_0\Gamma,\ C_3C_0\Gamma,\ B_iC_2C_3W^2,\ B_iC_3C_4W^2,\\ C_1C_5B_iW,\ C_3C_5B_iW,\ C_2C_4B_iW,\ C_1C_2C_3C_4B_iW,\\ C_iC_5B_iW^2,\ C_2C_0B_4W^2,\ C_4C_0B_2W^2\end{matrix}\right\},$$

where in (34) and (35), $\Gamma=I,\ B_3$; $i=2,4$. Now all the operators (31) occur in these three groups; all the operators (32) occur in the last two groups. No two of the three groups are conjugate within O since * G_{24}^3 contains G_8^3 self-conjugately, while L_{24} contains three groups conjugate with L_8, and T_{24} three groups conjugate with T_8.

To determine the Γ_{24} which contain G_{12}', we note that the operators of period 2 in H_{96} are those of G_{16} together with

$$(36)\qquad (\xi_1\xi_3)(\xi_r\xi_s)C\quad (C=I,\ C_1C_3,\ C_rC_s,\ C_1C_3C_rC_s;\ r,s=2,4,5).$$

Those of period 4 are given when C ranges over the remaining 12 operators of G_{16}. Now C_1C_3, C_1C_2, C_2C_3, B_3 and $C_1C_3B_3$ extend G_{12}' to the respective groups.

* There are exactly 7 operators of period 2 in G_{24}^3 and 9 in each L_{24} and T_{24}.

$$G''_{24} = \{\Gamma, \Gamma(\xi_2\xi_4\xi_5)^{\pm 1}, \quad (\Gamma = I, C_1C_5, C_2C_4, C_3C_0, C_2C_5, C_4C_5, C_2C_0, C_4C_0)\}, \tag{37}$$

$$G_{24} = \{\Gamma, \Gamma(\xi_2\xi_4\xi_5)^{\pm 1}, \quad (\Gamma = I, C_1C_2, C_1C_4, C_2C_4, C_1C_5, C_2C_5, C_4C_5, C_3C_0)\}, \tag{38}$$

$$\{\Gamma, \Gamma(\xi_2\xi_4\xi_5)^{\pm 1}, (\Gamma = I, C_2C_3, C_2C_4, C_3C_4, C_2C_5, C_3C_5, C_4C_5, C_1C_0)\}, \tag{39}$$

$$L^*_{24} = \{\Gamma, \Gamma(\xi_2\xi_4\xi_5)^{\pm 1}, \Gamma B_3, \Gamma(\xi_1\xi_3)(\xi_2\xi_5), \Gamma(\xi_1\xi_3)(\xi_4\xi_5), \quad (\Gamma = I, C_2C_4, C_2C_5, C_4C_5)\}. \tag{40}$$

$$\left\{\Gamma_1, \Gamma_1(\xi_2\xi_4\xi_5)^{\pm 1}, \Gamma_2 B_3, \Gamma_2(\xi_1\xi_3)(\xi_2\xi_5), \Gamma_2(\xi_1\xi_3)(\xi_4\xi_5), \quad \begin{pmatrix}\Gamma_1 = I, C_2C_4, C_2C_5, C_4C_5\\ \Gamma_2 = C_1C_3, C_5C_0, C_2C_0, C_4C_0\end{pmatrix}\right\}. \tag{41}$$

All the operators (36) lie in these five groups. Of the operators of period 4, the square of CB_3 is C_2C_4 if $C = C_2C_5$, C_4C_5, C_2C_0 or C_4C_0; the square of $C(\xi_1\xi_3)(\xi_2\xi_5)$ is C_2C_5 if $C = C_2C_4$, C_4C_5, C_2C_0 or C_5C_0; the square of $C(\xi_1\xi_3)(\xi_4\xi_5)$ is C_4C_5 if $C = C_2C_1$, C_2C_5, C_4C_0 or C_5C_0. But these 12 operators of period 4, whose squares belong to G'_{12} and hence extend G'_{12} to a Γ_{24}, belong to the groups (40) and (41). The squares of the remaining operators of period 4 in H_{96} do not belong to G'_{12} and hence such an operator of period 4 extends G'_{12} to a Γ_{48}. Finally, all the substitutions $\Gamma(\xi_2\xi_4\xi_5)^{\pm 1}$, Γ ranging over G_{16}, of period 3 or 6 in H_{96} belong to the groups (37)–(39).

Now B_3 transforms (39) into (38); C_1C_5 transforms (41) into (40).

23. THEOREM. *Within G, the group C_{24} is self-conjugate only under G_{72}.*

The group C_{24} defined by (14) is the direct product of the cyclic group $C_3 = (L_{1,1})$ by a group Γ_8 affecting only ξ_2 and η_2. By IV the latter is composed of the identity, one operator $T_{2,-1}$ of period 2, and 6 operators of period 4. Hence C_{24} contains a single cyclic subgroup C_6. But the latter is self-conjugate only under G_{72}, the direct product of C_3 and a binary group Γ_{24} having Γ_8 as a self-conjugate subgroup. Note that Γ_8 is of the type F'''_8.

24. THEOREM. *Within G, the group G^*_{24} is self-conjugate only under itself.*

Of the operators of G^*_{24}, every $P^iT_{2,-1}D$ is of period 4; P, P^5, $PT_{2,-1}$, $P^5T_{2,-1}$, $P^2T_{2,-1}$ and $P^4T_{2,-1}$ are of period 6; P^2 and P^4 are of period 3;

$$P^3, \quad T_{2,-1}, \quad P^3T_{2,-1}, \quad P^iD \quad (i = 0, 1, \cdots, 5),$$

are of period 2. Hence G^*_{24} contains exactly 3 cyclic subgroups of order 6, $C^*_6 = (P)$, $C''_6 = (P^2T_{2,-1})$, and $(PT_{2,-1})$. The latter is transformed into C''_6 by D, which transforms P into P^{-1}, and $T_{2,-1}$ into $P^3T_{2,-1}$. Now

C_6^* and C_6'' are not conjugate under G (§6). Hence an operator of G which transforms G_{24}^* into itself must transform C_6^* into itself and hence belong to G_{24}^* (§6).

25. THEOREM. *Within O, the group G_{24}^3 is self-conjugate only under G_{48}.*

The only subgroup of order 8 of G_{24}^3 is G_8^3 (§22). The latter is self-conjugate only under $H_{192} = (G_{64}, W)$ by III_{29}. Now B_2, C_1C_4 and $B_2C_1C_4$ transform B_2W into C_3C_4W, B_3W and $C_1C_2B_4W$, respectively, none of which belong to G_{24}^3. Hence the product of any operator of G_{48} (which transforms G_{24}^3 into itself by §22) by B_2, C_1C_4 or $B_2C_1C_4$ does not transform G_{24}^3 into itself. But ΓC_1C_4, ΓB_2, $\Gamma B_2C_1C_4$, where Γ ranges over G_{16}', give all the operators of G_{64} not in G_{16}'.

26. THEOREM. *Within O, L_{24} and T_{24} are self-conjugate only under G_{48}.*

The group L_{24} contains exactly 3 conjugate subgroups of order 8, one of which is L_8. The latter is self-conjugate only under G_{16}' by III_{35}. Hence at most $3 \cdot 16$ operators of O transform L_{24} into itself. But L_{24} is a subgroup of G_{48}. The proof for T_{24} follows by replacing L_{24}, L_8 by T_{24}, T_8.

27. THEOREM. *Within O, the group G_{24}'' is self-conjugate only under H_{96}.*

Indeed, by III_{29}, the self-conjugate subgroup G_8'' of G_{24}'' is self-conjugate only under H_{96}.

28. THEOREM. *Within O, the group L_{24}^* is self-conjugate only under* *

$$G_{48}'' = \{G_{16}'', (\xi_2\xi_4\xi_5)\}. \tag{42}$$

The group L_{24}^* defined by (40) contains 9 operators of period 2 and hence 3 subgroups of order 8. Now B_2 transforms L_{24}^* into $\{L_8, (\xi_1\xi_3\xi_5)\}$. But L_8 is self-conjugate only under G_{16}' by III_{35}. Now C_2C_4, which extends L_8 to G_{16}', transforms L_8 and $(\xi_1\xi_3\xi_5)$ each into itself. Hence $\{L_8, (\xi_1\xi_3\xi_5)\}$ is self-conjugate only under $\{G_{16}', (\xi_1\xi_3\xi_5)\}$, which is transformed into (42) by B_2.

29. THEOREM. *Within O, the group G_{24} is self-conjugate only under*

$$H_{48} = \{G_{16}, (\xi_2\xi_4\xi_5)\}. \tag{43}$$

The group G_{24} defined by (38) is transformed by $(\xi_1\xi_4)(\xi_3\xi_5)$ into

$$\{G_8, (\xi_1\xi_2\xi_3)\}. \tag{44}$$

Now G_8 is self-conjugate only under G_{192} by III_{30}. But the only even substitutions on $\xi_1, \xi_2, \xi_3, \xi_4$ which transform into itself the cyclic group generated by $(\xi_1\xi_2\xi_3)$ are the powers of the latter. Hence (44) is self-conjugate only under the group $\{G_{16}, (\xi_1\xi_2\xi_3)\}$ of order 48. Transforming it by $(\xi_1\xi_4)(\xi_3\xi_5)$, we obtain H_{48}.

* We readily see that the order of (42) is 48 and that it is a subgroup of H_{96}.

30. THEOREM. *Within O, a subgroup simply isomorphic with the group of order 24 generated by four operators subject to the generational relations*

$$A^4 = I,\ B^2 = A^2,\ B^{-1}AB = A^{-1},\ C^3 = I,\ C^{-1}AC = B,\ C^{-1}BC = AB$$

is conjugate with one of the three groups

$$(45)\quad F_{24} = \{F_8''', (\xi_2\xi_4\xi_3)\},\quad F'_{24} = \{F_8''', W\},\quad F^*_{24} = \{F_8''', (\xi_2\xi_4\xi_3)W\}.$$

As the subgroup generated by A and B we may take F_8'''. Now $(\xi_2\xi_4\xi_3)$, W, and each B_i transform F_8''' into itself. But $(\xi_2\xi_4\xi_3)$, $(\xi_2\xi_3\xi_4)$, and B_2 transform $B_2C_1C_4$ into $B_3C_1C_2$, $B_4C_1C_3$, and $B_2C_2C_3$, respectively; B_3 transforms $B_3C_1C_2$ into $B_3C_3C_4$; B_4 transforms $B_4C_1C_3$ into $B_4C_2C_4$. Hence we may take $A = B_2C_1C_4$. Next, B_4 is commutative with $B_2C_1C_4$, and transforms $B_4C_1C_3$ and $B_3C_1C_2$ into $B_4C_2C_4$ and $B_3C_3C_4$, respectively. Hence we may take $B = B_3C_1C_2$ or $B_4C_1C_3$. Since F_8''' is self-conjugate, C must leave ξ_5 fixed (III_{32}).

For $B = B_3C_1C_2$, the conditions $AC = CB$ and $BC = CAB$ give

$$C = \begin{Vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & 0 \\ -\alpha_{14} & \alpha_{13} & -\alpha_{12} & \alpha_{11} & 0 \\ -\alpha_{12} & \alpha_{11} & \alpha_{14} & -\alpha_{13} & 0 \\ -\alpha_{13} & -\alpha_{14} & \alpha_{11} & \alpha_{12} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{Vmatrix},$$

subject to the single condition $\alpha_{11}^2 + \alpha_{12}^2 + \alpha_{13}^2 + \alpha_{14}^2 \equiv 1$. If one of the α_{ij} is $\equiv 0$, then three are, the resulting substitutions of period 3 being

$$(\xi_2\xi_4\xi_3),\qquad (\xi_1\xi_2\xi_3)C_2C_3,\qquad (\xi_1\xi_3\xi_4)C_3C_4,\qquad (\xi_1\xi_4\xi_2)C_2C_4.$$

But $B_2C_1C_3$, $B_3C_2C_3$, $B_4C_1C_2$, which transform F_8''' into itself, transform $(\xi_2\xi_4\xi_3)$ into $(\xi_1\xi_3\xi_4)C_3C_4$, $(\xi_1\xi_4\xi_2)C_2C_4$, $(\xi_1\xi_2\xi_3)C_2C_3$.

Let next each $\alpha_{ij} \neq 0$ in C. The conditions for $C^2 = C^{-1}$ are

$$\alpha_{11} \equiv 1 - \alpha_{12}\alpha_{14} - \alpha_{12}\alpha_{13} - \alpha_{13}\alpha_{14},\qquad -\alpha_{14} \equiv -1 + \alpha_{11}\alpha_{12} + \alpha_{11}\alpha_{13} + \alpha_{12}\alpha_{13},$$

$$-\alpha_{12} \equiv -1 + \alpha_{11}\alpha_{13} + \alpha_{11}\alpha_{14} + \alpha_{13}\alpha_{14},\quad -\alpha_{13} \equiv -1 + \alpha_{11}\alpha_{12} + \alpha_{11}\alpha_{14} + \alpha_{12}\alpha_{14}.$$

Eliminating α_{11} from the last three, we get

$$(\alpha_{12}\alpha_{13} - 1)(\alpha_{14} + 1) \equiv 0,\qquad (\alpha_{13}\alpha_{14} - 1)(\alpha_{12} + 1) \equiv 0,$$

$$(\alpha_{12}\alpha_{14} - 1)(\alpha_{13} + 1) \equiv 0.$$

If $\alpha_{13} \equiv 1$, the conditions give merely $\alpha_{12} \equiv \alpha_{14}$, $\alpha_{11} = \alpha_{12}$. If $\alpha_{13} \equiv -1$, they reduce to $(\alpha_{12} + 1)(\alpha_{14} + 1) \equiv 0$, $\alpha_{11} = \alpha_{12}\alpha_{14}$. The resultant operators are

$$\begin{Vmatrix} \alpha & \alpha & 1 & \alpha & 0 \\ -\alpha & 1 & -\alpha & \alpha & 0 \\ -\alpha & \alpha & \alpha & -1 & 0 \\ -1 & -\alpha & \alpha & \alpha & 0 \\ 0 & 0 & 0 & 0 & 1 \end{Vmatrix}, \quad \begin{Vmatrix} \alpha_2\alpha_4 & \alpha_2 & -1 & \alpha_4 & 0 \\ -\alpha_4 & -1 & -\alpha_2 & \alpha_2\alpha_4 & 0 \\ -\alpha_2 & \alpha_2\alpha_4 & \alpha_4 & 1 & 0 \\ 1 & -\alpha_4 & \alpha_2\alpha_4 & \alpha_2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{Vmatrix}$$

$$[(\alpha_2+1)(\alpha_4+1)\equiv 0].$$

Call them $C_{(\alpha)}$ and C_{α_2,α_4}, respectively. Now $C_{-1,-1}=W$, so that we have the group (F_8''', W). Next $C_{-1,1}=C_1C_4WC_3C_4$, which C_1C_3 transforms into $WB_2C_1C_4$, an operator of (F_8''', W). Again, $C_{1,-1}=C_1C_2WC_2C_4$, which C_2C_3 transforms into $WB_3C_1C_2$, an operator of (F_8''', W). But W transforms F_8''' into itself and $C_{(-1)}$ into $C_{1,-1}$. Finally, $C_{(+1)}=W^2(\xi_2\xi_3\xi_4)$.

For $B=B_4C_1C_3$, the conditions $AC'=C'B$ and $BC'=C'AB$ give

$$C'=\begin{Vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & 0 \\ \alpha_{13} & \alpha_{14} & -\alpha_{11} & -\alpha_{12} & 0 \\ \alpha_{14} & -\alpha_{13} & \alpha_{12} & -\alpha_{11} & 0 \\ -\alpha_{12} & \alpha_{11} & \alpha_{14} & -\alpha_{13} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{Vmatrix},$$

subject to the single condition $\alpha_{11}^2+\alpha_{12}^2+\alpha_{13}^2+\alpha_{14}^2\equiv 1$. If one of the α_{ij} is $\equiv 0$, then three are, the resulting substitutions of period 3 being

$$(\xi_1\xi_4\xi_3), \quad (\xi_1\xi_2\xi_4)C_2C_4, \quad (\xi_1\xi_3\xi_2)C_1C_2, \quad (\xi_2\xi_3\xi_4)C_2C_3.$$

But the last three are the transforms of $(\xi_1\xi_4\xi_3)$ by $B_4C_1C_2$, $B_3C_2C_3$, $B_2C_1C_3$, respectively. The resulting group $[(F_8''', (\xi_1\xi_4\xi_3)]$ is transformed into F_{24} by B_2.

Let next each $\alpha_{ij}\not\equiv 0$ in C'. The conditions for $C''^2=C'^{-1}$ are

$$\alpha_{11}=1+\alpha_{12}\alpha_{13}-\alpha_{12}\alpha_{14}+\alpha_{13}\alpha_{14}, \qquad \alpha_{13}=-1+\alpha_{11}\alpha_{12}+\alpha_{11}\alpha_{14}+\alpha_{12}\alpha_{14},$$

$$\alpha_{14}=1-\alpha_{11}\alpha_{12}+\alpha_{11}\alpha_{13}+\alpha_{12}\alpha_{13}, \quad -\alpha_{12}=-1-\alpha_{11}\alpha_{13}+\alpha_{11}\alpha_{14}-\alpha_{13}\alpha_{14}.$$

Eliminating α_{11} from the last three, we get

$$(\alpha_{12}\alpha_{14}-1)(1-\alpha_{13})\equiv 0, \ (\alpha_{12}\alpha_{13}+1)(1+\alpha_{14})\equiv 0, \ (\alpha_{13}\alpha_{14}+1)(1+\alpha_{12})\equiv 0.$$

If $\alpha_{13}\equiv 1$, the conditions reduce to $(\alpha_{12}+1)(\alpha_{14}+1)\equiv 0$ and $\alpha_{11}=\alpha_{12}\alpha_{14}$. If $\alpha_{13}\equiv -1$, they reduce to $\alpha_{14}=\alpha_{12}$, $\alpha_{11}=\alpha_{12}$. The resulting operators are

$$\begin{vmatrix} \alpha & \alpha & -1 & \alpha & 0 \\ -1 & \alpha & -\alpha & -\alpha & 0 \\ \alpha & 1 & \alpha & -\alpha & 0 \\ -\alpha & \alpha & \alpha & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}, \quad \begin{vmatrix} \alpha_2\alpha_4 & \alpha_2 & 1 & \alpha_4 & 0 \\ 1 & \alpha_4 & -\alpha_2\alpha_4 & -\alpha_2 & 0 \\ \alpha_4 & -1 & \alpha_2 & -\alpha_2\alpha_4 & 0 \\ -\alpha_2 & \alpha_2\alpha_4 & \alpha_4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$[(\alpha_2+1)(\alpha_4+1)\equiv 0].$$

Call them $C'_{(\alpha)}$ and C'_{α_2,α_4} respectively. Now W transforms $C'_{(+1)}$ into $W(\xi_2\xi_4\xi_3)B_2C_2C_3$, which belongs to F''_{24}. Next, $C'_{(-1)}=W^2C_1C_4$, which C_1C_2 transforms into $B_3C_3C_4W^2$, an operator of F'_{24}. Next, $C'_{-1,-1}$ equals $W^2B_3C_1C_2C_3C_4$, which C_1C_3 transforms into W^2. Again, $C'_{1,-1}$ equals $W^2B_4C_2C_4$, which belongs to F_{24}. Finally,

$$C'_{-1,1}=W^2B_2C_3C_4,$$

which is transformed by C_1C_4 into $B_4C_2C_4W^2$, an operator of F'_{24}.

Each of these three contains F'''_8 self-conjugately and hence a single subgroup of order 8. Hence a group which transforms one of them into itself must transform F'''_8 into itself and therefore leave ξ_5 fixed. An operator of the latter is of the form W^aJ, where J belongs to J_{96}, merely permuting $\xi_1^2, \xi_2^2, \xi_3^2, \xi_4^2$. Hence it cannot transform W into an operator of F_{24}. Hence F_{24} and F'_{24} are not conjugate.

31. THEOREM. *Within O, F_{24} and F'_{24} are self-conjugate only under*

$$G'_{72}=\{F'''_8, (\xi_2\xi_4\xi_3), W\}. \tag{46}$$

Since $(\xi_2\xi_4\xi_3)$ and W are commutative, while each transforms F'''_8 into itself, the groups F_{24} and F''_{24} are each self-conjugate under G'_{72}.

To show that there are no further operators transforming F_{24} into itself, we note that C_1C_2 and C_3C_4, C_1C_3 and C_2C_4, C_1C_4 and C_2C_3, transform $(\xi_2\xi_4\xi_3)$ into $(\xi_2\xi_4\xi_3)C_2C_3$, $(\xi_2\xi_4\xi_3)C_3C_4$, $(\xi_2\xi_4\xi_3)C_2C_4$, respectively, while B_2 transforms $(\xi_2\xi_4\xi_3)$, C_2C_3, C_3C_4, C_2C_4 into $(\xi_1\xi_3\xi_4)$, C_1C_4, C_3C_4, C_1C_3, respectively. Hence $C_1C_4B_2$ and $C_2C_3B_2$ transform $(\xi_2\xi_4\xi_3)$ into $(\xi_1\xi_3\xi_4)C_1C_3$, while no other $C_iC_jB_2$ transforms $(\xi_2\xi_4\xi_3)$ into an operator of F_{24}. Now $C_1C_4B_2=B_2C_2C_3$ and $C_2C_3B_2=B_2C_1C_4$ belong to F'''_8. In this way it may be verified that every operator EC, where E is an even permutation of $\xi_1, \xi_2, \xi_3, \xi_4$ and C is a product of an even number of the C_i, must belong to F_{24} if it transforms $(\xi_2\xi_4\xi_3)$ into an operator of F_{24}.

To show that there are no operators other than those of G'_{72} which transform F'_{24} into itself, we note that B_2 and B_3 transform W into $B_2C_3C_4W$ and $B_3C_2C_4W$, respectively, neither of which occurs in F'_{24}. But B_2 extends

F_8''' to a group of order 16, which B_3 extends to G_{32}. In view of (47), F_{24}' is self-conjugate only under a group of order $\frac{1}{4}\cdot 288$.

32. THEOREM. *Within O, the group F_{24}^* is self-conjugate only under*

$$G_{288} = \{ G_{32}, (\xi_2\xi_3\xi_4), W \}. \tag{47}$$

Now F_{24}^* is self-conjugate under G_{72}. Moreover, B_2 and B_3 transform $(\xi_2\xi_4\xi_3)W$ into $B_2C_2C_3(\xi_2\xi_4\xi_3)W$ and $B_3C_3C_4(\xi_2\xi_4\xi_3)W$, respectively, each belonging to F_{24}^*. But B_2 and B_3 extend F_8''' to G_{32}.

The subgroups of order 36.

33. THEOREM. *Within G, the subgroups of order* 36 *are conjugate with one of the four: K_{36}^*, G_{36}''',*

$$K_{36}^{**} = (K_{18}^{**}, P_{12}), \qquad H_{36}^{**} = (H_{18}^{**}, T_{2,-1}W_0). \tag{48}$$

A Γ_{36} contains either a Γ_{18} or else a Γ_{12} of the non-cyclic commutative type.* Within G, the latter is self-conjugate only under a Γ_{24} (§ 9). *Hence a subgroup Γ_{36} of G contains a Γ_{18}.* For the latter we may take one of the 7 groups (23). But K_{18} is self-conjugate only under K_{54} (§ 14) and hence is excluded. The group G_{18}''' leads to G_{36}''' only (§ 20), while K_{18}^* and H_{18}^* lead to K_{36}^* only (§ 15, § 17).

Next, G_{18}^* and K_{18}^{**} are self-conjugate only under H_{108} (§ 16, § 18). The operators which extend one of these Γ_{18} to a subgroup Γ_{36} may be limited to the operators of period 2 of H_{108} or to those of period 6 whose squares belong to Γ_{18}. The former are conjugate within H_{108} with P_{12}, $P_{12}T_{2,-1}$, $T_{2,-1}$. Since P_{12} belongs to G_{18}^*, there results a single Γ_{36}:

$$(G_{18}^*, T_{2,-1}) = (K_9^*, P_{12}, T_{2,-1}) = K_{36}^*.$$

Since $T_{2,-1}$ belongs to K_{18}^{**}, there results only $(K_{18}^{**}, P_{12}) = K_{36}^{**}$. Consider next the operators of period 6 of H_{108} given at the end of § 4. Their squares are $[-k, 0, 0, -\gamma]$, $[\pm 1, 0, 0, \pm 1]$, and $[k, 0, -\gamma, k]$, respectively. The first two of these belong to G_{18}^*, while the third does only when $\gamma \equiv 0$. The resulting operators of period 6 are $[k, 0, 0, \gamma]T_{2,-1}$, $[\pm 1, 0, 0, 0]T_{2,-1}P_{12}$ and $[\pm 1, 0, 0, 0]P_{12}$; they either belong to G_{18}^* or extend it to $(G_{18}^*, T_{2,-1}) = K_{36}^*$. The only ones of the above squares which belong to $K_{18}^{**} = \{[-\gamma, 0, c, \gamma]T_{2,\pm 1}\}$ are $[\gamma, 0, 0, -\gamma]$ and $[0, 0, -\gamma, 0]$. The resulting operators of period 6 are $[-\gamma, 0, 0, \gamma]T_{2,-1}$ and $[0, 0, \gamma, 0]P_{12}$, where $\gamma \neq 0$, the first belonging to K_{18}^{**} and the second extending it to $(K_{18}^{**}, P_{12}) = K_{36}^{**}$.

Finally H_{18}^{**} is self-conjugate only under H_{216} (§ 19). The operators of period 2 of H_{216} are conjugate within it with $P_{12}T_{2,-1}$, $T_{2,-1}$ or W_0 (§ 5), the

* MILLER, Quarterly Journal, vol. 28 (1896), p. 283.

first belonging to H_{18}^{**}, the second and third extending H_{18}^{**} to K_{36}^{**} and G_{36}^{**}. The operators of period 4 are conjugate with $T_{2,-1}W_0$ (§ 5), which extends H_{18}^{**} to H_{36}^{**}. The operators of period 6 are conjugate with W_1, $[k, 0, 0, \gamma]T_{2,-1}$ or $[\pm 1, 0, 0, 0]T_{2,-1}P_{12}$ (§ 5). Their squares are $[-1, 0, 1, 1]$, $[-k, 0, -\gamma]$, $[\pm 1, 0, 0, \pm 1]$, the last not belonging to H_{18}^{**} and the second belonging to it only when $k = -\gamma$. The resulting operators $W_1 \equiv [1, 0, -1, -1]W_0$ and $[-\gamma, 0, 0, \gamma]T_{2,-1}$ extend H_{18}^{**} to $(H_{18}^{**}, W_0) = G_{36}^{**}$ and $(H_{18}^{**}, T_{2,-1}) = K_{36}^{**}$, respectively.

34. THEOREM. *Within G, the group K_{36}^{*} is self-conjugate only under itself.*

Since K_{36}^{*} is a subgroup of H_{108}, under which K_9^{*} is self-conjugate, K_9^{*} is the only group of order 9 contained in K_{36}^{*}, by SYLOW's theorem. Hence K_{36}^{*} is self-conjugate only under a subgroup of H_{108}. Since $[0, 0, 1, 0]$ extends K_9^{*} to K_{27}, it extends K_{36}^{*} to H_{108}; but it transforms $T_{2,-1}$ into $[0, 0, 1, 0]T_{2,-1}$, which is not in K_{36}^{*}.

35. THEOREM. *Within G, the group K_{36}^{**} is self-conjugate only under H_{216}.*

Since K_{36}^{**} is a subgroup of H_{216}, the largest group in which K_9^{**} is self-conjugate, K_9^{**} is the only group of order 9 in K_{36}^{**}. Hence K_{36}^{**} is self-conjugate only under H_{216} or a subgroup of it. Now $[1, 0, 0, 0]$ extends K_{36}^{**} to H_{108} and transforms P_{12} into $[-1, 0, 0, 1]P_{12}$, which belongs to K_{36}^{**}. Also, W_0 extends H_{108} to H_{216} and transforms P_{12} into $T_{2,-1}$. Hence K_{36}^{**} is certainly self-conjugate under H_{216}.

36. THEOREM. *Within G, the group G_{36}^{**} is self-conjugate only under*

$$G_{72}^{**} = (G_{36}^{**}, P_{12}) = (K_9^{**}, T_{2,-1}, P_{12}, W_0). \tag{49}$$

As in the preceding section, G_{36}^{**} is self-conjugate only under a subgroup of H_{216}. Now P_{12} transforms W_0 into $W_0T_{2,-1}P_{12}$ and hence extends G_{36}^{**} to a group of order 72. Now $[1, 0, 0, 0]$, which extends this G_{72}^{**} to H_{216}, transforms W_0 into $[0, 0, 1, 1]W_0$, which does not lie in G_{36}^{**}.

37. THEOREM. *Within G, the group H_{36}^{**} is self-conjugate only under G_{72}^{**}.*

As in § 35, H_{36}^{**} is self-conjugate only under a subgroup of H_{216}. Now W_0, which extends H_{108} to H_{216}, transforms $T_{2,-1}W_0$ into its inverse $W_0T_{2,-1}$. Hence H_{36}^{**} is certainly self-conjugate under $(H_{36}^{**}, W_0) = G_{72}^{**}$. The latter is extended to H_{216} by $[1, 0, 0, 0]$, which transforms $T_{2,-1}W_0$ into $T_{2,-1}[0, 0, 1, 1]W_0$ (§ 36). The latter equals $[0, 0, -1, 1]T_{2,-1}W_0$ and is not in H_{36}^{**} since $[0, 0, -1, 1]$ is not in K_9^{**}.

38. THEOREM. *No two of the groups K_{36}^{*}, K_{36}^{**}, G_{36}^{**}, H_{36}^{**} are conjugate within G.*

For the first three groups, the result follows from §§ 34–36. Neither K_{36}^{*} nor K_{36}^{**} contains operators of period 4, being subgroups of H_{108} (§ 4). The same is true for G_{36}^{**} since $(K_9^{**}, P_{12}T_{2,-1}) = H_{18}^{**}$ and $(K_9^{**}, W_0) = G_{18}^{**}$ have no operators of period 4, while

$$[-\gamma, 0, c, \gamma]\, P_{12}\, T_{2,-1}\, W_0 = \pm \begin{vmatrix} -1 & c+\gamma & 1 & \gamma-c \\ 0 & 1 & 0 & -1 \\ 1 & c-\gamma & 1 & c+\gamma \\ 0 & -1 & 0 & -1 \end{vmatrix}$$

is not of the form (3) if $\alpha_{22} \equiv +1$ and hence is not of period 4 (§5). But H_{36}^{**} contains $T_{2,-1} W_0$, of period 4.

The subgroups of order 48.

39. It is first shown that *every subgroup* Γ_{48} *contains a* Γ_{24}. A Γ_{48} not containing a Γ_{24} has * 16 cyclic Γ_3 and hence a self-conjugate Γ_{16}. The latter may be taken to be G_{16} or F_{16} by III. For $\Gamma_{16} = G_{16}$, Γ_{48} is a subgroup of G_{960} and hence is conjugate with H_{48} of § 29, which contains the substitution $(\xi_2\xi_4\xi_5) C_1 C_2$ of period 6. For $\Gamma_{16} = F_{16}$, Γ_{48} is a subgroup of G_{96}. Since G_{96} contains at most 16 conjugate Γ_3, all of them must belong to Γ_{48}. Hence the latter is $\{F_{16}, W(\xi_2\xi_4\xi_3)\}$, which contains $W(\xi_2\xi_4\xi_3) B_3$ of period 12, its cube being $B_2 C_2 C_4$.

We consider in turn for Γ_{24} the types of non-conjugate subgroups of order 24. Now C_{24}, F_{24}, F'_{24}, and G^*_{24} are excluded, being self-conjugate only under G_{72}, G'_{72}, G'_{72}, and G^*_{24}, respectively. Again, G^3_{24}, L_{24} and T_{24} lead only to G_{48}; G_{24} leads only to H_{48}; L^*_{24} leads only to G'_{48}.

The group G''_{24} is self-conjugate only under H_{96}. The operators of period 2 belonging to H_{96} but not to G''_{24} are given by (36) and

$$C_1C_2,\ C_1C_4,\ C_1C_5,\ C_2C_3,\ C_3C_4,\ C_3C_5,\ C_1C_6,\ C_3C_6. \tag{50}$$

Any one of the set (36) extends G''_{24} to G''_{48}, given by (42); any one of the set (50) extends G''_{24} to H_{48}, given by (43). Next, $C(\xi_1\xi_3)(\xi_r\xi_s)$, where C ranges over the operators of G_{16} other than I, C_1C_3, C_rC_s, $C_1C_3C_rC_s$, give the operators of period 4 of H_{96}. Their squares all belong to G''_{24}. Now $C_1C_2B_3$ extends G''_{24} to

$$H''_{48} = \{H''_{16}, (\xi_2\xi_4\xi_5)\}, \tag{51}$$

composed of the substitutions of G''_{24} and ΣB_3, $\Sigma(\xi_1\xi_3)(\xi_2\xi_5)$, $\Sigma(\xi_1\xi_3)(\xi_4\xi_5)$, where Σ ranges over the set (50). It contains all the CB_3 of period 4 except $C_2C_5B_3$, $C_4C_5B_3$, $C_2C_6B_3$, $C_4C_6B_3$, each of which extends G''_{24} to $(G''_{24}, B_3) = G''_{48}$; it contains all the $C(\xi_1\xi_3)(\xi_2\xi_5)$ of period 4 except for $C = C_2C_4$, C_4C_5, C_2C_6, C_5C_6; while, for these, $C(\xi_1\xi_3)(\xi_2\xi_5)$ extends G''_{24} to G''_{48}; likewise for the $C(\xi_1\xi_3)(\xi_4\xi_5)$. Finally, the operators of period 6 of H_{96} are the $\Gamma(\xi_2\xi_4\xi_5)^{\pm 1}$, where Γ ranges over the operators of G_{16} other than I,

* MILLER, Quarterly Journal, vol. 30 (1899), p. 245.

C_2C_4, C_2C_5, C_4C_5, which furnish operators of period 3 belonging to G'''_{24}. Hence the squares of those of period 6 belong to G''_{24}. Any $\Sigma(\xi_2\xi_4\xi_5)^{\pm 1}$, where Σ is one of the operators (50), extends G''_{24} to H_{48}; the remaining ones belong to G''_{24}.

Finally, F^*_{24} is self-conjugate only under G_{288}. To the latter corresponds (*Linear Groups*, § 189), the abelian group

$$(52) \qquad G'_{288} = \left\{ \pm \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a_1 & b_1 \\ 0 & 0 & c_1 & d_1 \end{bmatrix} \quad \begin{bmatrix} ad - bc \equiv 1 \\ a_1d_1 - b_1c_1 \equiv 1 \end{bmatrix} \right\},$$

since the second compound of the general operator (52) leaves Y_{12} and Y_{34} unaltered. By IV the binary group Γ_{24} has a single set of conjugate subgroups of each of orders 2, 3, 4, 6, 8, but no group of order 12. Hence every subgroup of order 48 of (52) is conjugate with $\pm(\Gamma_4, \Gamma_{24})$. Hence if a subgroup of G_{288} is of order 48 and contains F^*_{24}, it is conjugate with

$$(53) \qquad F_{48} = \{\Gamma, \Gamma(\xi_2\xi_4\xi_3)\,W, \Gamma W^2(\xi_2\xi_3\xi_4), (\Gamma \text{ ranging over } F_{16})\}.$$

The subgroups of order 48 *are conjugate with* G_{48}, H_{48}, G''_{48}, H''_{48} *or* F_{48}.

40. THEOREM. *Within* O, *the group* G_{48} *is self-conjugate only under itself.*

An operator which transforms G_{48} into itself must transform its 3 subgroups of order 16 amongst themselves. By III_{28}, G'_{16} is self-conjugate within O only under $J^3_{32} = \{G'_{16}, C_1C_2\}$. Now C_1C_2 transforms B_2W into $WC_1C_2C_3C_4$, which does not belong to G_{48}.

41. THEOREM. *Within* O, *the group* H_{48} *is self-conjugate only under* H_{96}.

By III, bottom of p. 20, H_{48} is self-conjugate only under a subgroup of G_{960}. But the only even substitutions on $\xi_1, \cdots, \xi_5$ which transform $(\xi_2\xi_4\xi_5)$ into itself or its inverse are the powers of $(\xi_2\xi_4\xi_5)$ and $(\xi_1\xi_3)(\xi_2\xi_4)$, $(\xi_1\xi_3)(\xi_2\xi_5)$, $(\xi_1\xi_3)(\xi_4\xi_5)$.

42. THEOREM. *Within* O, *the group* G''_{48} *is self-conjugate only under* H_{96}.

By the foot-note to § 28, G''_{48} is a subgroup of H_{96}. Now G''_{16} is one of 3 conjugates under G''_{48} and is self-conjugate only under a subgroup Γ_{32} of O.

43. THEOREM. *Within* O, *the group* H''_{48} *is self-conjugate only under* H_{96}.

By the method of § 42, the proof follows from § 39.

44. THEOREM. *Within* O, *the group* F_{48} *is self-conjugate only under*

$$(54) \qquad G_{96} = \{\Gamma, \Gamma(\xi_2\xi_4\xi_3)\,W, \Gamma W^2(\xi_2\xi_3\xi_4), (\Gamma \textit{ ranging over } G_{32})\}.$$

Indeed, F_{16} is self-conjugate under F_{48}, while within O it is self-conjugate only under G_{96} by III_{37}. For another proof, see end of § 39.

The subgroups of order 54.

45. THEOREM. *Within G, the groups of order* 54 *fall into* 3 *distinct sets of conjugate subgroups. As representatives, we may take*

$$(55)\quad G_{54}=(G_{27},\ T_{2,-1}),\qquad K_{54}=(K_{27},\ T_{2,-1}),\qquad K'_{54}=(K_{27},\ P_{12}T_{2,-1}).$$

They are self-conjugate only under the respective groups G_{648}, H_{108}, H_{216}.

A group of order 54 contains a self-conjugate group of order 27. The latter may be assumed to be H_{27}, K_{27} or G_{27}, which are self-conjugate within G only under G_{81}, H_{648}, G_{648}, respectively (II, pp. 378–380). Hence H_{27} is excluded.

The only operators of period 2 in G_{648} are $[0,\ a,\ c,\ 0]T_{2,-1}$, as shown directly or by § 50. Each is therefore (§ 13) conjugate with $T_{2,-1}$ within G_{648}, which has G_{162} as a subgroup.

In order that the general operator (19) of Π_{380} in H_{648} shall equal its inverse S^{-1} or $S^{-1}T_{1,-1}T_{2,-1}$, the conditions are, respectively,

$$\alpha_{11}\equiv\delta_{11},\quad \alpha_{22}\equiv\delta_{22},\quad \alpha_{21}\equiv\delta_{12},\quad \delta_{21}\equiv\alpha_{12},\quad \gamma_{22}\equiv\gamma_{11}\equiv 0,\ \gamma_{12}\equiv-\gamma_{21};$$

$$\alpha_{11}\equiv-\delta_{11},\ \alpha_{22}\equiv-\delta_{22},\ \alpha_{21}\equiv-\delta_{12},\ \delta_{21}\equiv-\alpha_{12},\ \gamma_{21}\equiv\gamma_{12}.$$

In the first case, the abelian conditions (see (19) of Π_{380}) give

$$\alpha_{11}^2+\alpha_{12}\delta_{12}\equiv\alpha_{22}^2+\alpha_{12}\delta_{12}\equiv 1,\qquad \alpha_{12}(\alpha_{11}+\alpha_{22})\equiv\delta_{12}(\alpha_{11}+\alpha_{22})\equiv\gamma_{12}(\alpha_{11}+\alpha_{22})\equiv 0,$$

so that the substitutions of period 2 are

$$\pm\begin{Vmatrix}1&0&\alpha_{12}&\gamma_{12}\\0&1&0&0\\0&-\gamma_{12}&-1&0\\0&\alpha_{12}&0&-1\end{Vmatrix},\qquad \pm\begin{Vmatrix}1&0&0&\gamma_{12}\\0&1&0&\delta_{12}\\\delta_{12}&-\gamma_{12}&-1&0\\0&0&0&-1\end{Vmatrix},$$

$$\pm\begin{Vmatrix}0&0&1&\gamma_{12}\\0&0&0&1\\1&-\gamma_{12}&0&0\\0&1&0&0\end{Vmatrix}.$$

The first is transformed into a like substitution Σ with $\alpha_{12}\equiv 0$ by

$$A_{\alpha_{12}}:\qquad \xi_1'=\xi_1-\alpha_{12}\xi_2,\qquad \eta_2'=\eta_2+\alpha_{12}\eta_1,$$

which belongs to H_{648}; while Σ is transformed into $T_{2,-1}$ by

$$B_{\gamma_{12}}:\qquad \xi_1'=\xi_1-\gamma_{12}\eta_2,\qquad \xi_2'=\xi_2-\gamma_{12}\eta_1,$$

likewise in H_{648}. Transforming the second by $A_{\delta_{12}}$, we obtain one of the third

type. The third type is transformed into P_{12} by $L_{2,\gamma_{12}}$, which belongs to H_{648}. But H_{648} contains Z_0 (§ 5) which transforms P_{12} into $T_{1,-1}$, identical with $T_{2,-1}$ in the quotient-group.

In the second case, the abelian conditions give

$$-\alpha_{11}^2+\alpha_{12}\delta_{12}\equiv-\alpha_{22}^2+\alpha_{12}\delta_{12}\equiv 1,\quad \alpha_{12}(\alpha_{11}+\alpha_{22})\equiv 0,\quad \delta_{12}\equiv(\alpha_{11}+\alpha_{22})\equiv 0,$$

so that the substitutions of period 2 are

$$\pm\begin{bmatrix}\alpha_{11} & \gamma_{11} & \alpha_{12} & \gamma_{12}\\ 0 & -\alpha_{11} & 0 & \delta_{12}\\ -\delta_{12} & \gamma_{12} & -\alpha_{11} & \gamma_{22}\\ 0 & -\alpha_{12} & 0 & \alpha_{11}\end{bmatrix},\qquad (\gamma_{11}\delta_{12}-\alpha_{11}\gamma_{12}+\alpha_{12}\gamma_{22}\equiv 0).$$

Transforming by $A_{-\alpha_{11}\delta_{12}}$, we obtain a similar substitution with $\alpha_{11}\equiv 0$. Transforming the latter by $B_{\gamma_{11}\alpha_{12}}$, we obtain a similar substitution with $\alpha_{11}\equiv\gamma_{11}\equiv 0$. Transforming by the substitution $L_{2,\alpha_{12}\gamma_{12}}$ of H_{648}, we have also $\gamma_{12}\equiv 0$. Hence $\gamma_{22}\equiv 0$, so that the final substitution is $P_{12}T_{2,-1}$. Since the latter has the characteristic determinant $(\rho^2+1)^2$, while the only operators of period 2 in K_{54}, viz., $[0,0,c,0]T_{2,-1}$, have the characteristic determinant $(\rho^2-1)^2$, the resulting groups K_{54} and K'_{54} are not conjugate.

To find the largest group in which K'_{54} is self-conjugate, we seek the operators (19) of Π_{380} which transform $P_{12}T_{2,-1}$ into one of the 9 operators $[-\gamma,0,c,\gamma]P_{12}T_{2,-1}$ of period 2. The conditions are

$$\alpha_{21}\equiv\pm\alpha_{12}\quad \alpha_{22}\equiv\mp\alpha_{11},\quad \delta_{21}\equiv\pm\delta_{12},\quad \delta_{22}\equiv\mp\delta_{11},$$

$$c\delta_{11}+\gamma\delta_{21}\equiv\pm\gamma_{12}-\gamma_{21},\quad \gamma\delta_{11}-c\delta_{21}\equiv\gamma_{11}\pm\gamma_{22}.$$

The latter serve merely to determine c and γ uniquely, since the determinant $\delta_{11}^2+\delta_{21}^2$ will be seen to be $\not\equiv 0$. The abelian conditions reduce to

$$\alpha_{11}\delta_{11}+\alpha_{12}\delta_{12}\equiv 1,\quad \alpha_{12}\delta_{11}-\alpha_{11}\delta_{12}\equiv 0,\quad \alpha_{11}\gamma_{21}+\alpha_{12}\gamma_{22}\mp\alpha_{12}\gamma_{11}\pm\alpha_{11}\gamma_{12}\equiv 0.$$

For $\alpha_{11}\equiv 0$, then $\alpha_{12}\equiv\delta_{12}$, $\delta_{11}\equiv 0$, $\gamma_{22}\equiv-\gamma_{11}$. For $\alpha_{11}\equiv 1$, either $\alpha_{12}\equiv\delta_{12}\equiv 0$, $\delta_{11}\equiv 1$, or $\alpha_{12}\equiv\pm 1$, $\delta_{12}\equiv\mp 1$, $\delta_{11}\equiv-1$. The resulting group is H_{216}.

For K_{54} we seek the operators (19) of Π_{380} which transform $T_{2,-1}$ into one of the operators $[0,0,c,0]T_{2,-1}$ of period 2. The conditions are

$$\alpha_{12}\equiv\alpha_{21}\equiv\delta_{12}\equiv\delta_{21}\equiv 0,\qquad \gamma_{12}\equiv c\delta_{22},\qquad \gamma_{21}\equiv c\delta_{11};$$

$$\alpha_{11}\equiv\alpha_{22}\equiv\delta_{11}\equiv\delta_{22}\equiv 0,\qquad \gamma_{11}\equiv c\delta_{21},\qquad \gamma_{22}\equiv c\delta_{12},$$

according as the upper or lower signs are taken. In the first case, the abelian conditions reduce to $\alpha_{11}\delta_{11}\equiv 1$, $\alpha_{22}\delta_{22}\equiv 1$, the resulting substitutions being the

U of § 3. In the second case, the abelian conditions reduce to $\alpha_{12}\delta_{12} \equiv 1$, $\alpha_{21}\delta_{21} \equiv 1$, the resulting substitutions being the V of § 3. The U and the V form H_{108}.

The subgroups of order 60.

46. THEOREM. *The subgroups of order* 60 *of* O *fall into two distinct sets of conjugates, represented by* G_{60} *of* III_3 *and* G'_{60}, *which are self-conjugate* only under* G_{120} *and* G'_{120}, *respectively, where*†

$$(56)\quad G'_{60} = \{(\xi_1\xi_2\xi_3\xi_4\xi_5),\ Q\},\qquad G_{120} = \{G_{60},\ \Sigma\},\qquad G'_{120} = \{G'_{60},\ \Sigma\},$$

Q *and* Σ *being respectively*

$$(57)\quad \begin{Vmatrix} -1 & -1 & -1 & 0 & 1 \\ -1 & 0 & -1 & -1 & 1 \\ -1 & -1 & 0 & -1 & 1 \\ 0 & -1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{Vmatrix}, \quad \begin{Vmatrix} -1 & -1 & -1 & -1 & 0 \\ -1 & -1 & 0 & -1 & -1 \\ 0 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 0 & -1 \\ -1 & 0 & -1 & -1 & -1 \end{Vmatrix}.$$

Since no subgroup of order a divisor of < 60 of 60 is self-conjugate in a Γ_{60} by the earlier results (see the table), a subgroup Γ_{60} must be simple and hence simply isomorphic with the alternating group $G^{(5)}_{60}$ on 5 letters.‡

Within O, all the operators of period 5 are conjugate by I. Hence we may assume that Γ_{60} contains the linear substitution $A = (\xi_1\xi_2\xi_3\xi_4\xi_5)$. The cyclic group G_5 generated by it is seen (§ 21) to be self-conjugate only under

$$(58)\quad G'_{20} = \{A, \Sigma\} = \{A^i, A^i\Sigma, A^i\Sigma^2, A^i\Sigma^3 (i = 0, 1, 2, 3, 4)\}.$$

Since $\Sigma^2 = (\xi_1\xi_2)(\xi_3\xi_5)$, the only operators of period 2 in G'_{20} are

$$(\xi_1\xi_2)(\xi_3\xi_5),\quad (\xi_1\xi_3)(\xi_4\xi_5),\quad (\xi_1\xi_4)(\xi_2\xi_3),\quad (\xi_1\xi_5)(\xi_2\xi_4),\quad (\xi_2\xi_5)(\xi_3\xi_4),$$

which are transformed into each other transitively by the powers of A. They belong to Γ_{60} since it contains 5 operators of period 2 which transform G_5 into itself. Now Γ_{60} is simply isomorphic with the abstract group generated by A_1 and B where

$$(59)\qquad A_1^5 = B^2 = (A_1 B)^3 = I.$$

*Cf. Proceedings of the London Mathematical Society, vol. 31 (1899), p. 53.

†Note that $\Sigma = W^2(\xi_1\xi_2\xi_3\xi_4\xi_5)\, W^2(\xi_1\xi_5\xi_4\xi_2\xi_3)\, C_1C_2C_3C_4$, $Q = C_3C_4WC_2C_5(\xi_1\xi_5)(\xi_2\xi_4)\Sigma^{-1}$, so that Σ and Q belong to O.

‡To give another proof, a Γ_{60} contains 1 or 6 conjugate Γ_5. But a Γ_5 is self-conjugate only under a Γ_{20} within O. Hence a subgroup Γ_{60} contains 6 conjugate Γ_5 and is simply isomorphic with $G^{(5)}_{60}$ (BURNSIDE, *The Theory of Groups*, pp. 107–108).

These relations are satisfied when $A_1 = A$, $B = (\xi_1\xi_2)(\xi_3\xi_4)$, whence $(\xi_1\xi_4)(\xi_2\xi_3)B$ becomes $(\xi_1\xi_3)(\xi_2\xi_4)$ of period 2. Hence the normalized Γ_{60}, which contains A and $(\xi_1\xi_4)(\xi_2\xi_3)$, may be generated by A and a substitution B such that B, $(\xi_1\xi_4)(\xi_2\xi_3)B$ and AB are of periods 2, 2 and 3, respectively. Now the condition that an orthogonal substitution shall be of period 2 is that its matrix be symmetrical with respect to the main diagonal. Hence B and $(\xi_1\xi_4)(\xi_2\xi_3)B$ are both of period 2 if and only if

$$B = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} \\ \alpha_{12} & \alpha_{22} & \alpha_{23} & \alpha_{13} & \alpha_{25} \\ \alpha_{13} & \alpha_{23} & \alpha_{22} & \alpha_{12} & \alpha_{25} \\ \alpha_{14} & \alpha_{13} & \alpha_{12} & \alpha_{11} & \alpha_{15} \\ \alpha_{15} & \alpha_{25} & \alpha_{25} & \alpha_{15} & \alpha_{55} \end{bmatrix}.$$

The orthogonal conditions on the 1st and 4th rows, 2d and 3d rows, give

$$(60) \qquad -\alpha_{11}\alpha_{14} - \alpha_{12}\alpha_{13} + \alpha_{15}^2 \equiv 0, \qquad -\alpha_{12}\alpha_{13} - \alpha_{22}\alpha_{23} + \alpha_{25}^2 \equiv 0.$$

The orthogonal condition on the last row gives $\alpha_{55}^2 - \alpha_{15}^2 - \alpha_{25}^2 \equiv 1$, whence

$$\alpha_{55} \equiv \pm 1,\ \alpha_{15} \equiv \alpha_{25} \equiv 0; \qquad \text{or} \qquad \alpha_{55} \equiv 0,\ \alpha_{15} \neq 0,\ \alpha_{25} \neq 0.$$

In the first case, we find that

$$(AB)^{-1} = \begin{bmatrix} 0 & \cdots & \pm 1 \\ \alpha_{11} & \cdots & 0 \\ \alpha_{12} & \cdots & 0 \\ \alpha_{13} & \cdots & 0 \\ \alpha_{14} & \cdots & 0 \end{bmatrix}, \quad (AB)^2 = \begin{bmatrix} \pm\alpha_{14} & \cdots & \alpha_{12}^2 - \alpha_{11}\alpha_{13} \\ \pm\alpha_{13} & \cdots & \alpha_{12}\alpha_{13} + \alpha_{12}\alpha_{22} + \alpha_{11}\alpha_{23} \\ \pm\alpha_{12} & \cdots & \alpha_{13}^2 + \alpha_{12}\alpha_{23} + \alpha_{11}\alpha_{22} \\ \pm\alpha_{11} & \cdots & \alpha_{13}\alpha_{14} + \alpha_{12}\alpha_{13} + \alpha_{11}\alpha_{12} \\ 0 & \cdots & \pm\alpha_{14} \end{bmatrix}.$$

Equating these when the lower signs are taken, we get $\alpha_{14} \equiv 0$, $\alpha_{13} \equiv -\alpha_{11}$, $\alpha_{12} \equiv 0$, $\alpha_{11}\alpha_{13} \equiv 1$, which are contradictory since -1 is a quadratic non-residue of 3. For the upper signs,

$$\alpha_{14} \equiv 0,\ \alpha_{13} \equiv \alpha_{11},\ \alpha_{12}^2 - \alpha_{11}\alpha_{13} \equiv 1,\ \alpha_{12}\alpha_{13} + \alpha_{12}\alpha_{22} + \alpha_{11}\alpha_{23} \equiv 0,$$
$$\alpha_{13}^2 + \alpha_{12}\alpha_{23} + \alpha_{11}\alpha_{22} \equiv 0.$$

Hence $\alpha_{12}^2 - \alpha_{11}^2 \equiv 0$, so that $\alpha_{11} \equiv \alpha_{13} \equiv 0$, $\alpha_{12} \neq 0$, $\alpha_{22} \equiv 0$, $\alpha_{23} \equiv 0$. Hence

$$B = (\xi_1\xi_2)(\xi_3\xi_4) \qquad \text{or} \qquad (\xi_1\xi_2)(\xi_3\xi_4)C_1C_2C_3C_4.$$

The latter is excluded since its product by $(\xi_1\xi_2\xi_3\xi_4\xi_5)$ on the left is not of period 3.

In the second case ($\alpha_{55} \equiv 0$), we equate the coefficients in the first and fifth rows of the matrices for $(AB)^{-1}$ and $(AB)^2$, and get

$\alpha_{15} \equiv 1 + \alpha_{11}\alpha_{25} + \alpha_{12}\alpha_{25} + \alpha_{13}\alpha_{15}$, $\quad \alpha_{25} \equiv \alpha_{11}\alpha_{15} + \alpha_{11}\alpha_{12} + \alpha_{12}\alpha_{13} + \alpha_{13}\alpha_{14} + \alpha_{14}\alpha_{15}$,

$\alpha_{14} \equiv 1 - \alpha_{15}\alpha_{25}$, $\quad \alpha_{13} \equiv 1 + \alpha_{12}\alpha_{15} + \alpha_{13}\alpha_{25} + \alpha_{14}\alpha_{25}$, $\quad 0 \equiv \alpha_{12}^2 - \alpha_{11}\alpha_{13} - \alpha_{14}\alpha_{15}$,

$\alpha_{25} \equiv \alpha_{13}^2 + \alpha_{12}\alpha_{15} + \alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{23} + \alpha_{14}\alpha_{25}$, $\quad \alpha_{12} \equiv \alpha_{22}\alpha_{15} + \alpha_{23}\alpha_{25} + \alpha_{13}\alpha_{25} + \alpha_{15}\alpha_{25}$,

$\alpha_{15} \equiv \alpha_{13}\alpha_{15} + \alpha_{11}\alpha_{23} + \alpha_{12}\alpha_{22} + \alpha_{12}\alpha_{13} + \alpha_{14}\alpha_{25}$, $\quad \alpha_{11} \equiv \alpha_{23}\alpha_{15} + \alpha_{22}\alpha_{25} + \alpha_{12}\alpha_{25} + \alpha_{15}\alpha_{25}$.

From the first, third and fourth it follows that

$$\alpha_{12} \equiv 1 - \alpha_{15} - \alpha_{15}\alpha_{25} + \alpha_{13}\alpha_{15} - \alpha_{13}\alpha_{15}\alpha_{25}, \quad \alpha_{11} \equiv -1 + \alpha_{15} - \alpha_{25} - \alpha_{15}\alpha_{25} - \alpha_{13}\alpha_{15}.$$

Eliminating α_{11}, α_{12}, α_{14} from the second condition of the preceding set and from the first relation (60), we get, respectively,

$$\alpha_{13}^2(\alpha_{15} - 1)(1 - \alpha_{25}) + \alpha_{13}(\alpha_{15}\alpha_{25} + \alpha_{15} - \alpha_{25} + 1) - \alpha_{25} \equiv 0,$$

$$\alpha_{13}^2(\alpha_{15}\alpha_{25} - \alpha_{15}) + \alpha_{13}(\alpha_{15}\alpha_{25} - \alpha_{15} - \alpha_{25} - 1) + 1 + \alpha_{15} - \alpha_{25} \equiv 0.$$

Eliminating α_{13}^2, we get $(\alpha_{13} + \alpha_{25})(\alpha_{15} + 1) \equiv 0$. If $\alpha_{15} \equiv -1$, the second becomes

$$\alpha_{13}^2(1 - \alpha_{25}) + \alpha_{13}\alpha_{25} - \alpha_{25} \equiv 0.$$

By trial, α_{13} is neither 0 nor -1. Hence $\alpha_{13} \equiv 1$, $\alpha_{25} \equiv 1$. Then $\alpha_{14} \equiv -1$, $\alpha_{11} \equiv -1$, $\alpha_{12} \equiv 0$. The last four of the above set of 9 conditions then reduce to $\alpha_{22} \equiv \alpha_{23} \equiv -1$. But the resulting substitution AB is not of period 3. Hence must $\alpha_{15} \equiv +1$, $\alpha_{13} \equiv -\alpha_{25}$. Hence $\alpha_{25} \equiv +1$, $\alpha_{13} \equiv -1$. Then $\alpha_{11} \equiv -1$, $\alpha_{12} \equiv -1$, $\alpha_{14} \equiv 0$. The last four of the above set of 9 conditions then reduce to $\alpha_{22} + \alpha_{23} \equiv -1$. For these values of α_{ij}, the necessary and sufficient condition that AQ shall have period 3 is $\alpha_{22} \equiv 0$. The resulting substitution B is Q given by (57).

Since G_5 is self-conjugate only under G_{20}, while Γ_{60} contains but 6 conjugate groups of order 5, it follows that at most 120 operators of O transform Γ_{60} into itself. That G_{60} is self-conjugate under G_{120} follows from the fact that Σ, an operator of period 4 defined by (57), transforms $(\xi_1\xi_2\xi_3\xi_4\xi_5)$ into its square, and $(\xi_1\xi_2)(\xi_3\xi_4)$ into $(\xi_2\xi_4)(\xi_3\xi_5)$. That G'_{60} is self-conjugate under G'_{120} follows since Σ transforms Q into $(\xi_1\xi_5)(\xi_2\xi_4)Q(\xi_5\xi_4\xi_3\xi_2\xi_1)$, an operator of G'_{60}.

Finally, G_{60} and G'_{60} are not conjugate within O. If an operator T of O transforms G'_{60} into G_{60} and any subgroup G'_5 of G'_{60} into G_5 and if S be one of the existing operators of G'_{120} which transforms G'_{60} into itself and G'_5 into G_5, then $S^{-1}T$ transforms G_5 into itself and G'_{60} into G_{60}. Hence $S^{-1}T$ belongs to G_{20}, a subgroup of G'_{120}. Hence T belongs to G'_{120} and transforms G'_{60} into itself, contrary to hypothesis.

The subgroups of order 72.

47. Every Γ_{72} contains 1 or 4 conjugate Γ_9. If a Γ_9 is self-conjugate, the quotient-group Γ_{72}/Γ_9 is a group of order 8 having a subgroup of order 4, so

that Γ_{72} has a subgroup of order 36. Let next Γ_g be one of 4 conjugate subgroups of Γ_{72}. The group Γ_g which transforms into itself each of the four is self-conjugate under Γ_{72} and g is a divisor, less than 18, of 18. Evidently $g \geqq 3 = 72/4!$, and $g \neq 9$. If $g = 6$, Γ_g must be cyclic, since a non-cyclic subgroup of order 6 of G is self-conjugate only under a group of order 36 or 108 (§7). Moreover, the cyclic Γ_6 are self-conjugate only under subgroups conjugate with G_{72}, K_{36}^*, or G_{24}^* (§6). Hence we may take $\Gamma_6 = C_6$, $\Gamma_{72} = G_{72}$. Finally, if $g = 3$, Γ_{72}/Γ_g is simply isomorphic with the symmetric group on 4 letters, so that Γ_{72} contains a subgroup of order 36. *Hence Γ_{72} is conjugate with G_{72} or else contains a subgroup of order* 36.

The subgroup K_{36}^* is excluded, since it is self-conjugate only under itself (§34). Each of the groups G_{36}^{**} and H_{36}^{**} leads to G_{72}^{**} only (§§36, 37). Finally, K_{36}^{**} is self-conjugate only under H_{216} (§35). A substitution which extends K_{36}^{**} to a Γ_{72} must be an operator of H_{216} of period 2, 4, or 6, whose square belongs to K_{36}^{**} (§5). For those of period 2, we may take $P_{12}T_{2,-1}$, $T_{2,-1}$ or W_0 (§5), of which the first two belong to K_{36}^{**} and are excluded. But W_0 extends K_{36}^{**} to G_{72}^{**}. Those of period 4 are conjugate with $T_{2,-1}W_0$, whose square $P_{12}T_{2,-1}$ belongs to K_{36}^{**}; it extends the latter to G_{72}^{**}. As at the end of §33, we may restrict the operators of period 6 to $W_1 = [1, 0, -1, -1]W_0$ and $[-\gamma, 0, 0, \gamma]T_{2,-1}$, the latter belonging to K_{36}^{**} and the former extending it to G_{72}^{**}. *Hence Γ_{72} is conjugate with G_{72}^{**} if it contains a subgroup of order* 36.

Since $T_{2,-1}$ is the only operator of period 2 in G_{72}, an operator which transforms G_{72} into itself must be commutative with $T_{2,-1}$ and hence by I_{109} be one of the operators A or AP_{12}, where A is an operator (52). Considering the binary substitutions on ξ_1 and η_1, we must have

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \qquad (ad - bc \equiv 1),$$

k to be suitably chosen. Hence $c \equiv 0$, $a \equiv kd$. Since $ad \equiv 1$, we have $a \equiv d \equiv \pm 1$, $k \equiv 1$. Placing the ambiguity of sign in front of the matrix for A, we may take $a \equiv d \equiv 1$. Hence A belongs to G_{72}. But P_{12} does not transform G_{72} into itself. *Hence G_{72} is self-conjugate only under itself.*

Since G_{72}^{**} is a subgroup of H_{216}, the largest subgroup of G in which K_9^{**} is self-conjugate, K_9^{**} is the only subgroup of order 9 of G_{72}^{**}. Hence the latter is self-conjugate only under a subgroup of H_{216}. Now $[1, 0, 0, 0]$, which extends G_{72}^{**} to H_{216}, transforms W_0 into $[0, 0, 1, 1]W_0$, which does not belong to G_{72}^{**}. *Hence G_{72}^{**} is self-conjugate only under itself.*

That G_{72} and G_{72}^{**} are not isomorphic follows from a consideration of their operators of period 2 or of their subgroups of order 9.

THEOREM. *Within G there are exactly two distinct sets of conjugate subgroups of order* 72. *As representatives we may take G_{72} and G_{72}^{**}; each is self-conjugate only under itself.*

The subgroups of order 80.

48. A Γ_{80} contains 1 or 16 conjugate Γ_5, the first alternative being here excluded (§ 2). Having 64 operators of period 5, Γ_{80} contains a single Γ_{16}, which is therefore self-conjugate. Hence Γ_{16} is conjugate with G_{16} (III). If G_{16} be taken as Γ_{16}, Γ_{80} must be a subgroup of G_{960} by III, bottom of p. 20. Every operator of period 5 in the latter is of the form PC, where C belongs to G_{16} and P is a cyclic permutation of $\xi_1, \cdots, \xi_5$. Within G_{960}, P is conjugate with $(\xi_1\xi_2\xi_3\xi_4\xi_5)$ or its inverse. Hence Γ_{80} is conjugate with

$$G_{80} = \{ G_{16}, (\xi_1\xi_2\xi_3\xi_4\xi_5) \}. \tag{61}$$

A substitution commutative with G_{80} must be commutative with G_{16} and hence belong to G_{960}. Of the even substitutions on $\xi_1, \cdots, \xi_5$ only the powers of $S = (\xi_1\xi_2\xi_3\xi_4\xi_5)$ transform the latter into itself, none transform S into either S^2 or S^3, while $(\xi_2\xi_5)(\xi_3\xi_4)$ transforms S into S^4.

THEOREM. *Within G, every subgroup of order* 80 *is conjugate with G_{80}. The latter is self-conjugate only under*

$$G_{160} = \{ G_{16}, (\xi_1\xi_2\xi_3\xi_4\xi_5), (\xi_2\xi_5)(\xi_3\xi_4) \}. \tag{62}$$

Subgroups of order 96.

49. Any Γ_{96} contains 1 or 3 conjugate Γ_{32}. In the first case, we may take G_{32} as Γ_{32}, so that Γ_{96} is a subgroup of G_{576}, the group of all substitutions replacing ξ_5 by $\pm\xi_5$. An operator of period 3 of G_{576} must replace ξ_5 by $+\xi_5$ and hence belong to $G_{288} = [G_{32}, W, (\xi_2\xi_3\xi_4)]$. We may therefore limit Γ_{96} to the groups obtained by extending G_{32} by respectively $W, (\xi_2\xi_3\xi_4), W(\xi_2\xi_3\xi_4)$, $W(\xi_2\xi_4\xi_3)$. But C_1C_5 transforms G_{32} into itself and $W(\xi_2\xi_3\xi_4)$ into $W^2(\xi_2\xi_3\xi_4)$, the inverse of the last extender. The three remaining groups are G_{96} of § 44 and

$$J_{96} = [G_{32}, (\xi_2\xi_3\xi_4)], \qquad L_{96} = (G_{32}, W). \tag{63}$$

Since each has a single subgroup of order 32, an operator transforming one of them into itself or another must transform G_{32} into itself and hence belong to G_{576}. Now G_{96}, J_{96} and L_{96} are each self-conjugate in G_{288}. Moreover, C_1C_5 transforms W into W^2, $(\xi_2\xi_3\xi_4)$ into itself, and G_{32} into itself. *Hence J_{96} and L_{96} are each self-conjugate only under the subgroup G_{576} of O, and are not conjugate within O; but G_{96} is self-conjugate only under G_{288}.*

Let next Γ_{96} contain 3 conjugate Γ_{32}. Let Γ_g be the group of the operators of Γ_{96} which transform each Γ_{32} into itself. Then g is a divisor of 32, $g < 32$, $g \geqq 96/3!$. Hence $g = 16$. Since Γ_{16} is self-conjugate in Γ_{96}, Γ_{16} is conjugate with F_{16} or G_{16}. Taking $\Gamma_{16} = F_{16}$, we have $\Gamma_{96} \equiv G_{96}$. Taking $\Gamma_{16} = G_{16}$, we have Γ_{96} as a subgroup of G_{960}. Within the latter, Γ_{96} is conjugate with

H_{96}, defined by (22). We first transform one of the operators of period 3 of G_{60} into $(\xi_2\xi_4\xi_5)$ and proceed as in § 41.

By III, bottom of p. 20, H_{96} is self-conjugate only under a subgroup of G_{960}. But the only even substitutions on $\xi_1, \cdots, \xi_5$ which transform $(\xi_1\xi_3)(\xi_2\xi_4)$, $(\xi_1\xi_3)(\xi_2\xi_5)$, $(\xi_1\xi_3)(\xi_4\xi_5)$ amongst themselves are these three and the powers of $(\xi_2\xi_4\xi_5)$. *Hence H_{96} is self-conjugate only under itself within O.*

The Subgroups of order 108.

50. Any Γ_{108} contains 1 or 4 conjugate Γ_{27}. Let first Γ_{27} be self-conjugate. Then Γ_{108}/Γ_{27} has a subgroup of order 2, so that Γ_{108} has a self-conjugate subgroup Γ_{54}. For the latter we may take G_{54}, K_{54} or K'_{54} (§ 45). But K_{54} leads to H_{108} only (§ 45).

Next, G_{54} is self-conjugate (§ 45) only under the group G_{648} of the operators (2) of Π_{372}. The square of (2) is*

$$\pm\begin{bmatrix} 1 & -k-\gamma a^2+\beta c^2+ac(\alpha-\delta) & a+a\alpha+c\beta & c+c\delta+a\gamma \\ 0 & 1 & 0 & 0 \\ 0 & (1+\alpha)(\alpha c-\gamma a)+\gamma(\beta c-\delta a) & \alpha^2+\beta\gamma & \gamma(\alpha+\delta) \\ 0 & (1+\delta)(\beta c-\delta a)+\beta(\alpha c-\gamma a) & \beta(\alpha+\delta) & \delta^2+\beta\gamma \end{bmatrix} \quad (a\delta-\beta\gamma\equiv 1).$$

This operator belongs to G_{54} only when $\alpha \equiv \delta$, $\beta \equiv \gamma \equiv 0$, whence (2) belongs to G_{54}, or $\alpha \equiv -\delta$, $\alpha^2+\beta\gamma \equiv \pm 1$, so that $\pm$ must be $-$. But there exists (IV) a binary substitution of determinant unity which transforms $\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & -\alpha\end{smallmatrix}\right)$ into $\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$. Hence, in the second case, (2) is conjugate within G_{648} with $[k, a, c, 0]M_2$, where M_2 replaces ξ_2 by η_2, and η_2 by $-\xi_2$. Hence every Γ_{108} defined by G_{54} is conjugate with

$$G_{108} = (G_{54}, M_2). \tag{64}$$

Finally, K'_{54} is self-conjugate only under H_{216} (§ 45). Within H_{216} the operators of period 2 are conjugate with $P_{12}T_{2,-1}$, $T_{2,-1}$ or W_0 (§ 5), the first belonging to K'_{54}, the second and third extending it to respectively H_{108} and

$$K'_{108} = (K'_{54}, W_0) = (K_{27}, P_{12}T_{2,-1}, W_0). \tag{65}$$

The operators of period 4 of H_{216} are conjugate with $T_{2,-1}W_0$ (§ 5), whose square $P_{12}T_{2,-1}$ belongs to K'_{54}, so that it extends K'_{54} to

$$K''_{108} = (K'_{54}, T_{2,-1}W_0) = (K_{27}, P_{12}T_{2,-1}, T_{2,-1}W_0). \tag{66}$$

Consider lastly the operators of period 6 given in the theorem of § 5. Their squares (§ 33, end) all belong to K_{27}, so that none are excluded. But

* Hence (2) is of period 2 only if $\beta \equiv \gamma \equiv 0$, $\alpha \equiv \delta \equiv -1$, $k \equiv 0$, giving $[0, a, c, 0]T_{2,-1}$.

$W_1 = [1, 0, -1, -1]\, W_0$ extends K'_{54} to K'_{108}, $[k, 0, 0, \gamma]\, T_{2,-1}$ extends K'_{54} to H_{108}, while $[\pm 1, 0, 0, 0]\, T_{2,-1} P_{12}$ belongs to K'_{54}.

We pass now to the case in which Γ_{27} is one of 4 conjugate subgroups. The group Γ_g of the operators of Γ_{108} which transform each of the four Γ_{27} into itself is self-conjugate under Γ_{108} and g is a divisor < 27 of 27, while $g \equiv 108/4!$. Hence $g = 9$, so that Γ_9 is conjugate with K''_9 or K'''_9 (Π_{386}). In the first case, Γ_{108} is conjugate with H_{108}. In the second case, Γ_{108} is conjugate with a subgroup of H_{216}. Since Γ_{108}/Γ_9 is simply isomorphic with the alternating group on 4 letters, which contains a self-conjugate group of order 4, Γ_{108} contains a self-conjugate group of order 36. By §§ 34–37, this must be of the type K'''_{36}, which is self-conjugate under H_{216}. Applying a suitable transformation within H_{216}, we may assume that Γ_{108} contains also K_{27}, with which all the subgroups of order 27 of H_{216} are conjugate, so that $\Gamma_{108} = H_{108}$.

Since H_{108} contains a single group K_{27} of order 27, and since K_{27} is self-conjugate only under H_{648}, H_{108} is self-conjugate only under a subgroup of H_{648}. But H_{108} is self-conjugate under H_{648}.

Since G_{108} contains a single group G_{27} of order 27 and since G_{27} is self-conjugate only under G_{648}, G_{108} is self-conjugate only under a subgroup of G_{648}. The latter affects ξ_2 and η_2 by a binary group of 24 substitutions $\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)$ of determinant unity. Within the latter, the group $\{I, T_{2,-1}, M_2, T_{2,-1} M_2\}$ is self-conjugate only under a group of order 8 (IV). Hence G_{108} is self-conjugate only under

(67) $G_{216} = \{\text{operators (2) of } \Pi_{372} \text{ with } \alpha \equiv \delta \equiv \pm 1,\ \beta \equiv \gamma \equiv 0, \text{ or else } \alpha + \delta \equiv 0\}$.

Finally, K'_{108} and K''_{108} are self-conjugate only under H_{216}.

Within G, the subgroups of order 108 *are conjugate with H_{108}, G_{108}, K'_{108} or K''_{108}. These are self-conjugate only under H_{648}, G_{216}, H_{216}, H_{216}, respectively.*

The subgroups of order 120.

51. THEOREM.* *Within G, every subgroup of order* 120 *is conjugate with G_{120} or G'_{120}. The latter are self-conjugate only under themselves.*

Every Γ_{120} is composite. By the earlier results (see table), a Γ_{120} has no self-conjugate subgroup of order < 60. We may therefore assume that it contains G_{60} or G'_{60}, whence Γ_{120} is G_{120} or G'_{120}, respectively (§ 46).

*Second proof: A Γ_{120} contains 1 or 6 conjugate Γ_5, the first case being here excluded. Since Γ_{120} thus contains a Γ_{20} but no self-conjugate subgroup lying in Γ_{20}, it is (DYCK) expressible as a transitive substitution-group on 6 letters and hence is simply isomorphic with the symmetric group on 5 letters. A third proof depends upon the 5 or 15 conjugate Γ_8, there being no primitive group of order 120 on 15 letters (MILLER, 2); while of the two imprimitive groups, one has substitutions of period 15 and is excluded, and the other is isomorphic with $G^{(5)}_{120}$ (KUHN).

Exclusion of the order 144.

52. Every Γ_{144} is composite.* By the earlier results, a group of order a divisor <144 of 144 is self-conjugate in a group of order a multiple of 144 only when it is conjugate with G_2, F_8''', F_{24}^*, or G_{32}. Of these, F_8''' and F_{24}^* are self-conjugate only under G_{288}, which has no subgroup of order 144 (see end of §39). Again, G_2 and G_{32} are self-conjugate only under G_{576}, which corresponds to the abelian group (A, P_{12}), A being defined by (52). Since the group G'_{288} of the A has no subgroup of index 2, Γ_{144} must contain $A'P_{12}$ and exactly one fourth of the operators A, the latter consequently (IV) forming a group given by the direct product (Γ_6, Γ_{24}) of a Γ_6 on one pair of the variables ξ_i, η_i and a Γ_{24} on the other pair. But $A'P_{12}$ transforms (Γ_6, Γ_{24}) into a group (Γ_{24}, Γ_6). Hence Γ_{144} does not exist.

The subgroups of order 160.

53. A Γ_{160} contains 1 or 5 conjugate Γ_{32}. By III, a subgroup Γ_{32} of G is self-conjugate only under a Γ_{64} or a Γ_{576}. The group Γ_g transforming each of the 5 Γ_{32} into itself is self-conjugate under Γ_{160} and g is a divisor <32 of 32. But by III no subgroup of order 2, 4, or 8 is self-conjugate under a Γ_{160}. Hence $g=16$ and Γ_{16} is conjugate with G_{16}, Γ_{160} with a subgroup of G_{960}. Proceeding as in § 48, we find that Γ_{160} is conjugate with G_{160}.

The 5 subgroups of order 32 of G_{160} are conjugate within G with J_{32}^k of III_5, which is self-conjugate only under G_{64}. Hence G_{160} is transformed into itself by at most 320 operators of G. (This also follows from the fact that G_{160} contains 16 conjugate Γ_5.) But G contains no Γ_{320} (§ 59).

THEOREM. *Within G, every subgroup of order* 160 *is conjugate with G_{160}, which is self-conjugate only under itself.*

The subgroups of order 162.

54. A Γ_{162} contains a single Γ_{81}. The latter may be taken to be G_{81} of II_{372}, whence the former is G_{162} of II_{373}, since G_{81} is self-conjugate only under G_{162}.

THEOREM. *Within G, every subgroup of order* 162 *is conjugate with G_{162}, which is self-conjugate only under itself.*

* The proof may be modified so as not to assume that Γ_{144} is composite. It has 1, 3, or 9 conjugate Γ_{16}, the first case being excluded. If Γ_{16} is one of 3 conjugates the largest group Γ_g transforming each into itself is of order 24 and hence conjugate with F_{24}^* (§§22–32). If Γ_{16} is one of 9 conjugates, g is a divisor <16 of 16. If $g=8$, Γ_g is conjugate with F_8'''. Also $g \neq 4$ by III. If $g=2$, Γ_2 is conjugate with $\{I, T_{2,-1}\}$, so that Γ_{144} is a subgroup of (A, P_{12}). If $g=1$, Γ_{144} is simply isomorphic with a transitive substitution-group on 9 letters. Such a group contains substitutions of period 8 (COLE, 1) and is here excluded.

Exclusion of the order 180.

55. A Γ_{180} is composite.* By the table, no subgroup of order $\leqq 90$ of G is self-conjugate in a group of order divisible by 180.

The subgroups of order 192.

56. THEOREM. *Within O, there are exactly two distinct sets of conjugate subgroups of order* 192, *represented by G_{192} and H_{192}; G_{192} is self-conjugate only under itself, and H_{192} only under G_{576}.*

We may take G_{64} as one of the conjugate subgroups of order 2^6 in Γ_{192}. If G_{64} is self-conjugate, $\Gamma_{192} = G_{192}$ by III_{21}. Next, let G_{64} be one of 3 conjugate subgroups. The group Γ_g of the operators of Γ_{192} which transform into itself each of the three groups of order 64 is self-conjugate under Γ_{192}. Moreover, $g < 64$ and $g \geqq 192/3!$, so that $g = 32$. Hence Γ_g is conjugate with G_{32}. Taking $\Gamma = G_{32}$, we have Γ_{192} as a subgroup of G_{576}. Under the latter, G_{64} has exactly 3 conjugate subgroups: G_{64}, $W^{-1}G_{64}W$, $W^{-2}G_{64}W^2$. Hence $\Gamma_{192} = (G_{64}, W) = H_{192}$.

A substitution which transforms G_{192} into itself must belong to G_{960} by III, bottom of p. 20. But the even substitutions on $\xi_1, \cdots, \xi_5$ which transform G_{192} into itself do not alter ξ_5. Hence G_{192} is self-conjugate only under itself.

A substitution S which transforms H_{192} into itself must replace ξ_5 by $\alpha_{5s}\xi_s$, $s \geqq 5$, by III_{20}. Likewise, S^{-1} must replace ξ_5 by $\alpha_{5t}\xi_t$, $t \leqq 5$, so that S replaces ξ_t by $\pm\xi_5$. Hence $S = S'(\xi_5\xi_s\cdots\xi_t)$, where S' leaves ξ_5 and ξ_s unaltered except in sign. Then $s = 5$; for if not, $S' = CS''$, where C is a product of the C_i, and S'' permutes the variables other than ξ_5 and ξ_s, so that S transforms $(\xi_1\xi_3)(\xi_2\xi_4)$ into a substitution permuting certain variables including ξ_5 and hence not in H_{192}. It has now been shown that S belongs to G_{576}. Inversely, $(\xi_2\xi_3\xi_4)$ extends H_{192} to G_{576} and transforms H_{192} into itself.

The subgroups of order 216.

57. A Γ_{216} has 1 or 4 conjugate Γ_{27}. Let first Γ_{27} be self-conjugate. Since Γ_{216}/Γ_{27} has a subgroup of order 4, Γ_{216} contains a Γ_{108}. Since the latter contains Γ_{27} self-conjugately, it is conjugate with H_{108}, G_{108}, K'_{108} or K''_{108} (§ 50). The last three lead only to G_{216} and H_{216} (§ 51, end). If $\Gamma_{108} = H_{108}$, Γ_{216} is a subgroup of H_{648}. But H_{648}/K_{27} is simply isomorphic with the symmetric group

* We may proceed otherwise. A Γ_{180} contains 1, 4, or 10 conjugate Γ_9. By II, the subgroups of order 9 are self-conjugate only under groups of orders 27, 108, 162, 216. Hence a subgroup Γ_{180} contains 10 conjugate Γ_9. The group Γ_g transforming each into itself is self-conjugate under Γ_{180} and $g < 18$. As before $g \neq 9$. Also, $g \neq 6$, $g \neq 3$, $g \neq 2$ by §§ 6, 7, 3 and III. Hence Γ_{180} is simply isomorphic with a transitive substitution-group on 10 letters. But no such group exists (COLE, 2).

on 4 letters, whose subgroups of order 8 are all conjugate (V). Hence Γ_{216} is conjugate with H_{216}, and H_{216} is self-conjugate only under itself within G. Let next Γ_{27} be one of 4 conjugate subgroups. The group Γ_g transforming each into itself is self-conjugate under Γ_{216} and g is a divisor < 54 of 54, while $g \equiv 216/4!$. Hence $g = 9, 18$, or 27. If $g = 9$, Γ_g is conjugate with K_9'' and Γ_{216} with H_{216} (II_{383}). If $g = 18$, Γ_g has a single Γ_9, consequently self-conjugate under Γ_{216}. If $g = 27$, we are led to the first case.

THEOREM. *The subgroups of order* 216 *are conjugate with* G_{216} *or* H_{216}, *the former being self-conjugate only under* G_{648} *and the latter only under itself.*

The subgroups of order 288.

58. Since * a Γ_{288} has a self-conjugate subgroup of order $\leqq 144$, it is conjugate with G_{288} or with a subgroup of G_{576} (by the table). In the latter case also, it is conjugate with G_{288} (§ 52).

By III_{20}, a substitution which transforms G_{288} into itself belongs to G_{576}.

THEOREM. *Every subgroup of order* 288 *of* G *is conjugate with* G_{288}, *which is self-conjugate only under* G_{576}.

Exclusion of the order 320.

59. A Γ_{320} contains 1 or 5 conjugate Γ_{64}. But a subgroup Γ_{64} of G is self-conjugate only under a Γ_{192} (III_{21}). Hence Γ_{64} is one of 5 conjugates. The group Γ_g transforming each into itself is self-conjugate under Γ_{320} and g is a divisor < 64 of 64, $g \equiv 320/5!$. Now the values 4, 8, 32 of g are excluded by III. For $g = 16$, we may take $\Gamma_{16} = G_{16}$, whence Γ_{320} is a subgroup of G_{960}. Then Γ_{320}/Γ_{16} would be simply isomorphic with a subgroup of the alternating group on 5 letters (III_3).

The subgroups of order 324.

60. A Γ_{324} has 1 or 4 conjugate Γ_{81}, one of which may be taken to be G_{81}. The latter, being self-conjugate only under G_{162}, is one of 4 conjugate subgroups of Γ_{324}. If Γ_g be the group transforming each of the 4 into itself, g is a divisor < 81 of 81 and $g \equiv 324/4!$. Hence $g = 27$. We may take $\Gamma_{27} = G_{27}$, or K_{27}, whence Γ_{324} is a subgroup of G_{648} or H_{648}, respectively. But G_{648}/G_{27} has

* For a second proof, we note that a Γ_{288} contains 1, 3, or 9 conjugate Γ_{32}. If Γ_{32} is self-conjugate it is conjugate with G_{32}, and Γ_{288} with a subgroup of G_{576} (III). If one of 3 conjugates, the group transforming each into itself is of order 48, so that Γ_{288}/Γ_{48} has a subgroup of order 3, and Γ_{288} a subgroup Γ_{144}, contrary to § 52. If Γ_{32} is one of 9 conjugates, g is a divisor < 32 of 32. But $g = 16$ and $g = 4$ are excluded by III; $g = 8$ leads to a group conjugate with F_8''', whence Γ_{288} is conjugate with G_{288}. For $g = 2$, we may take $\Gamma_2 = \{I, T_{2,-1}\}$, whence $\Gamma_{288} = G_{288}$ (§ 47). If $g = 1$, Γ_{288} is simply isomorphic with a transitive substitution group on 9 letters, which is impossible (COLE, 1).

no subgroup of index 2. However H_{648}/K_{27} contains a (single) subgroup of order 12 (V). Hence Γ_{324} is conjugate with

$$(68) \qquad H_{324} = \{\text{operators (19) of } \Pi_{380} \text{ with } \delta_{11}\delta_{22} - \delta_{12}\delta_{21} \equiv 1\}.$$

The number of operators of G commutative with H_{324} is evidently 4×162.

THEOREM. *Every subgroup of order* 324 *of* G *is conjugate with* H_{324}, *which is self-conjugate only under* H_{648}.

The subgroups of order 360.

61. A subgroup Γ_{360} has no self-conjugate subgroup of order $\leqq 180$ in view of the earlier results (see the table). Hence it is simply isomorphic with the alternating group G^6_{360} and contains subgroups of order 60. Now $(\xi_1\xi_4)(\xi_5\xi_6)$ extends G^5_{60} to G^6_{360}; its product by $(\xi_1\xi_4)(\xi_2\xi_3)$ on the left is of period 2; its product by $(\xi_1\xi_2\xi_3\xi_4\xi_5)$ on the left is of period 3. Hence we may assume that Γ_{360} has the normalized subgroup Γ_{60} of § 46 and an operator B of period 2 such that $(\xi_1\xi_4)(\xi_2\xi_3)B$ is of period 2 and $(\xi_1\xi_2\xi_3\xi_4\xi_5)B$ of period 3. Hence $B = (\xi_1\xi_2)(\xi_3\xi_4)$ or Q (§ 46).

THEOREM. *Within* O, *every subgroup of order* 360 *is conjugate with*

$$(69) \qquad G_{360} = \{G_{60}, G'_{60}\} = \{(\xi_1\xi_2\xi_3\xi_4\xi_5), (\xi_1\xi_2)(\xi_3\xi_4), Q\}.$$

The simple group G_{360} *is self-conjugate only under*

$$(70) \qquad G_{720} = \{G_{360}, \Sigma\}.$$

Exclusion of the order 432.

62. A Γ_{432} is composite. But no subgroup of order < 432 is self-conjugate in a group of order divisible by 432 (see table).*

*Second proof. A Γ_{432} has 1, 4, or 16 conjugate Γ_{27}. The first case is excluded by III. Let first Γ_{27} be one of 4 conjugates. The group Γ_g transforming each into itself is self-conjugate under Γ_{432} and g is a divisor < 108 of 108, while $g \geqq 432/4! = 18$. But $g \neq 27$ as before, $g \neq 18$ by §§ 14–20, $g \neq 36$ by §§ 33–37, $g \neq 54$ by § 45. Let next Γ_{27} be one of 16 conjugates. Then g is a divisor < 27 of 27. But $g \neq 9$ by II, $g \neq 3$ by § 3. If $g = 1$, Γ_{432} is simply isomorphic with a transitive group on 16 letters. Having no self-conjugate Γ_{16} by III, it is not primitive (MILLER, 3, p. 229). It is not imprimitive in view of the following theorem and proof communicated to me August 22 by Professor G. A. MILLER: *There exists no imprimitive group of degree* 16 *and of order* 432, 864, 1296, *or* 2592. For, an imprimitive group of degree 16 must have 8, 4, or 2 systems of imprimitivity, which are permuted according to a transitive group of degree 8, 4, or 2, respectively. The latter group may be assumed primitive. Since a primitive group of degree 8 has its order divisible by 7, the imprimitive groups under consideration would have either 4 or 2 systems. In the former case, these systems would be permuted according to G^4_{12} or G^4_{24}. The head would be divisible by 9, which is impossible since its order could not be divisible by 144. Hence the required group must contain just 2 systems of imprimitivity and hence have an intransitive subgroup of index 2 with constituents of degree 8. This is impossible since the order of the head would be divisible by 27 so that each constituent of the head would have an order divisible by 9. The order of such a constituent would therefore be divisible by 32, so that the order of the entire group would be divisible by 64.

Exclusion of the order 480.

63. A composite* Γ_{480} may be assumed (from the table) to contain G_{16} and to be a subgroup of G_{960}, whereas G_{960}/G_{16} is simply isomorphic with $G_{60}^{(5)}$ and has no subgroup of order 30.

The subgroups of order 576.

64. A subgroup Γ_{576}, being composite, must be conjugate† with G_{576} by the table of the earlier results. Now G_{576} contains exactly 3 groups conjugate with G_{64}, since the latter is self-conjugate only under G_{192}, a subgroup of G_{576} (III_{21}).

THEOREM. *Every subgroup of order* 576 *of* O *is conjugate with* G_{576}, *which is self-conjugate only under itself.*

The subgroups of order 648.

65. We may assume that Γ_{648} contains G_{81}, which is self-conjugate only under G_{162} (II_{373}). Hence Γ_{648} contains 4 groups conjugate with G_{81}. The group Γ_g transforming each of the 4 into itself is self-conjugate under Γ_{648} and g is a divisor <162 of 162, $g \geqq 648/4!$. Hence $g = 81$, 54, or 27. As above, $g \neq 81$. If $g = 54$, Γ_{648} is conjugate with G_{648} (§ 45). If $g = 27$, Γ_{648} is conjugate with G_{648} or H_{648} (II_{379}, II_{380}). Evidently the largest subgroup of G which transforms Γ_{648} into itself is of order 4×162.

THEOREM. *The subgroups of order* 648 *of* G *are conjugate with* G_{648} *or* H_{648}, *each of which is self-conjugate only under itself.*

The subgroups of order 720.

66. A subgroup Γ_{720}, necessarily composite, contains no self-conjugate subgroup of order <360 by the table of the earlier results. We may thus assume that Γ_{720} contains G_{360}, so that it is identical with G_{720} (§ 61).

THEOREM.‡ *Every subgroup of order* 720 *of* O *is conjugate with* G_{720}, *which is self-conjugate only under itself.*

It may be shown that G_{720} is simply isomorphic with the symmetric group on 6 letters.‡

* Second proof. A Γ_{480} has 1, 6, 16, or 96 conjugate Γ_5; 1, 4, 10, 16, 40, or 160 conjugate Γ_3. But a subgroup Γ_5 of G is self-conjugate only under a Γ_{20}, a Γ_3 only under a group of order a divisor of 648. Hence Γ_{480} contains 384 operators of period 5 and either 80 or 320 operators of period 3, and 32 further operators of a Γ_{32}.

† Second proof. A Γ_{576} has 1, 3, or 9 conjugate Γ_{64}, the first case being excluded. If 3 conjugates, the group Γ_g transforming each into itself is of order 96, so that Γ_{576} is conjugate with G_{576} (§ 49). If 9 conjugates, g is a divisor <64 of 64. If $g = 32$ or 2, Γ_{576} is conjugate with G_{576} by III. Also, $g \neq 16$, $g \neq 8$, $g \neq 4$. Finally, $g \neq 1$, since there is no transitive $G_{576}^{(9)}$ (COLE, 1).

‡ Cf. Proceedings of the London Mathematical Society, vol. 31 (1899), pp. 30–68; ser. 2, vol. 1 (1903–4), pp. 283–4.

Exclusion of the orders 810 *and* 864.

67. Subgroups of these orders are excluded * by the table, being composite.

The subgroups of order 960.

68. A subgroup Γ_{960}, being composite, must be conjugate † with G_{960} by the table. Now G_{960} is self-conjugate only under itself by III, bottom of p. 20.

Exclusion of the orders 1080, 1296, 1620, *and* 1728.

69. Subgroups of these orders are composite, ‡ but contain no self-conjugate subgroups (table).

* To give another proof for 810, we may assume that Γ_{810} contains G_{81}, which is self-conjugate only under G_{162}. Hence Γ_{810} contains 10 groups conjugate to G_{81}. But no subgroup of O of order 27, 9, or 3 is self-conjugate in a Γ_{810} by II. Hence Γ_{810} is simply isomorphic with a transitive substitution-group on 10 letters, whereas no such group exists (COLE, 2).

To give another proof for 864, we note that Γ_{864} contains 1, 4, or 16 conjugate Γ_{27}, the first case being excluded by II. Let first there be 4 conjugates and denote by Γ_g the largest subgroup of Γ_{864} transforming each into itself. Then g is a divisor of 216, $g \geqq 36 = 864/4!$. The resulting values 216, 108, 72, 54, 36 are excluded by the earlier results. Suppose next there are 16 conjugates. Then g is a divisor < 54 of 54. But 27, 18, 9, 6, 3, 2 are excluded by the earlier results. The case $g = 1$ is excluded by MILLER 3, and the foot-note to § 62.

† Another proof follows from a consideration of the 5 or 15 conjugate subgroups of order 64, there being no primitive group $G_{960}^{(15)}$ (MILLER, 2) and no such imprimitive group (KUHN).

‡ Second proof for 1080. A Γ_{1080} contains 1, 6, 36, or 216 conjugate Γ_5. But a subgroup Γ_5 is self-conjugate only under a Γ_{20}. Hence Γ_5 is one of 216 conjugates, so that Γ_{1080} contains 864 operators of period 5. It contains 1, 4, 10, or 40 conjugate Γ_{27}, the first two cases being excluded by II. Let first there be 10 conjugates. No group of order g, where $1 < g \leqq 108$, is self-conjugate under a Γ_{1080} by the table. Moreover, there is no transitive group $G_{1080}^{(10)}$ (COLE, 2). Hence there are 40 conjugate Γ_{27}. If they are of either of the types G_{27}, K_{27}, they form a complete set of conjugates under G_{25920}. Now the group K_{27} of the $[k, 0, c, \gamma]$ contains $L_{1,1} = [1, 0, 0, 0]$, $L_{1,1} L_{2,1} = [1, 0, 0, 1]$, $L_{1,-1} L_{2,1} = [-1, 0, 0, 1]$, so that the conjugates to K_{27} contain $2(40 + 120 + 240)$ operators of period 3. The group G_{27} of the $[k, a, c, 0]$ contains $L_{1,1}$ and also $[-1, 0, -1, 0]$, into which $L_{1,-1} L_{2,1} = [-1, 0, 0, 1]$ is transformed by the abelian substitution

$$\xi_1' = \xi_1, \ \eta_1' = \eta_1 - \eta_2, \ \xi_2' = \xi_2 + \xi_1, \ \eta_2' = \eta_2.$$

Hence the conjugates to G_{27} contain at least $2(40 + 240)$ operators of period 3. Hence the conjugate Γ_{27} are of neither of the types G_{27}, K_{27}, and hence of the type H_{27}. By II_{379}, the subgroups of order 9 of H_{27} are the commutative K_9 and three conjugate cyclic C_9. Within G_{25920}, each C_9 is self-conjugate only under H_{27} by II_{385}. Hence a C_9 is one of 40 distinct conjugates within Γ_{1080}, so that the latter contains $6 \times 40 = 240$ operators of period 9. But there were only 216 operators of period $\neq 5$.

Second proof for 1296. A Γ_{1296} contains 1, 4, or 16 conjugate Γ_{81}, the first two cases being excluded by II. The largest subgroup of Γ_{1296} which transforms each of the 16 Γ_{81} into itself is of order 1 by II. But there exists no primitive G_{1296}^{16} having no self-conjugate Γ_{16} (MILLER, 3, p. 229). There exists no imprimitive G_{1296}^{16} by the foot-note to § 62.

Second proof for 1620 (the lengthy case of JORDAN, *Traité*, pp. 322-9). A Γ_{1620} contains 1, 4, or 10 conjugate Γ_{81}. But a subgroup Γ_{81} is self-conjugate only under a G_{162}. Hence there are 10 conjugates. Now Γ_{1620} has no self-conjugate subgroup of order 2, 3, 6, 9, 18, or 27, and hence none of order 54. But there exists no transitive group $G_{1620}^{(10)}$ (COLE, 2).

Second proof for 1728. A subgroup of G_{25920} of index 15 must be maximal (§ 2), and hence requires the existence of a primitive $G_{25920}^{(15)}$, contrary to MILLER, 2.

Exclusion of the orders 2160, 2592, 2880.

70. If a subgroup of one of these orders exists, it must be maximal (§ 2). But there exists no primitive group of order 25920 on 12 letters (MILLER, 1), nor on 10 letters * (COLE, 2), nor on 9 letters (COLE, 1).

Maximal subgroups of G_{25920}.

71. THEOREM. *Every maximal subgroup of* G_{25920} *is conjugate with one of the five groups:* G_{960}, G_{720}, G_{648}, H_{648}, G_{576}.

By the table the only subgroups self-conjugate only under themselves are the above five and the following non-maximal ones: G_{20} (in G_{720}), G_{24}^{*} (in H_{648}, since its generators $P_{12}\,L_{1,-1}\,T_{1,-1}$, $T_{2,-1}$, and D are all of the form (19) of Π_{380}), K_{36}^{*} (in H_{108}), G_{48} (in G_{576}, since its substitutions replace ξ_5 by $\pm\xi_5$), G_{72} (in G'_{288} of § 39), H_{96} (in G_{960}), G_{120} and G'_{120} (in G_{720}), G_{160} and G_{192} (in G_{960}), G_{162} and H_{212} (in H_{648}).

THE UNIVERSITY OF CHICAGO,
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* Second proof for 2592. A Γ_{2592} contains 1, 4, or 16 conjugate Γ_{81}, the first two cases being here excluded. But a G_{2592}^{16} is neither primitive (MILLER, 3) nor imprimitive (foot-note to § 62).

ON THE INVARIANTS OF QUADRATIC DIFFERENTIAL FORMS, II*

BY

CHARLES NELSON HASKINS

In the following paper I propose to extend to differential parameters the work of an earlier article † on the determination of the number of differential invariants of quadratic differential forms in n variables.

§ 1. *Introduction.*

The form in question,

$$\phi \equiv \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik}(x_1, \cdots, x_n) dx_i dx_k,$$

and the transformation to which it is subjected,

$$Xf \equiv \sum_{r=1}^{n} \xi_r(x_1, \cdots, x_n) \frac{\partial f}{\partial x_r},$$

are the same as in the earlier paper.

A *differential parameter* of the form ϕ is a function which is unaltered by any transformation of the group, and which may involve the following quantities:

a) The coefficients a_{ik} and their derivatives of various orders with respect to the variables x;

b) certain arbitrary functions $U(x_1, \cdots, x_n)$, numerically unchanged by the group (2), and the various derivatives of these functions with respect to the variables x;

c) if the variables x are not all independent, the derivatives of those which are dependent with respect to those which are independent.

If the differential parameter does not contain the functions U, or their derivatives, or the derivatives of the x's, it will be called a *differential invariant* (Gaussian invariant).

If the differential parameter does not contain the derivatives of the x's it will be called a *differential parameter of the first type* (Beltramian invariant).

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† HASKINS, *On the invariants of quadratic differential forms.* Transactions of the American Mathematical Society, vol. 3 (1902), p. 71. (Cited hereafter as *Invariants* I.)

If the differential parameter contains the derivatives of the x's, but does not contain the U's or their derivatives, it will be called a *differential parameter of the second type* (Minding invariant).

The *order μ of a differential invariant* is the order of the highest derivative of any of the a_{ik}'s appearing in it.

The *order μ of a differential parameter* is the order of the highest derivative of any of the U's or x's appearing in it.

We shall find that differential *parameters* of a given order μ are naturally associated with differential *invariants* of the next lower order $\mu - 1$.

Differential parameters of the second type are always expressible in terms of differential parameters of the first type. *

For suppose the x's are not independent, but are bound by relations

$$U_i(x_1, \cdots, x_n) = 0 \qquad (i = 1, 2, \cdots, n-m),$$

and let F be a differential parameter of the second type. By means of the equations $U = 0$ and their derivatives with respect to m of the variables x, taken as independent, we may express in terms of the derivatives of the functions U the derivatives of the dependent x's with respect to the independent x's. When these expressions are substituted in F it becomes a function of the partial derivatives of the U's. It has the property of invariance and is therefore a parameter of the *first type*.

Geometrically, parameters of the second type express invariant properties of spaces of lower dimensions drawn in the space whose linear element is given by ϕ, e. g., invariant properties of curves on surfaces.

Not all parameters of the first type are also of the second type. Hence it is desirable to investigate the parameters of the second type as well as those of the first.

The differential parameters are solutions of certain complete systems of linear partial differential equations. The number of parameters is therefore equal to the excess of the number of variables in these equations over the number of independent equations. The computation of the number of equations and of the number of variables is very simple. Consequently the difficulty of the problem lies in the determination of the independence of the equations in question. It will be found that this determination can, save in a few cases, be referred to the corresponding problem in the case of the invariants. In fact there remain but five cases which cannot be so referred, viz.:

$$n = 2, \qquad \mu = 1, 2, 3;$$

$$n > 2, \qquad \mu = 1, 2.$$

* This point, apparently unnoticed by ŻORAWSKI, has recently been emphasized by FORSYTH, Philosophical Transactions, vol. 201 (1903), A, p. 333.

The cases $n = 2$ have been fully treated by ŻORAWSKI,* and may be omitted here. We have to consider, then, only the cases $n > 2$; $\mu = 1, 2$. It may be noted that LEVI-CIVITA has obtained a lower limit for the number of parameters.†

The consideration of the cases $n > 2$, $\mu = 1, 2$, and $\mu > 2$ occupies the last five sections of the article. The equations determining the parameters are so numerous and complicated that, in order to prove their independence, it is found desirable to introduce certain new variables, in particular, the three index symbols of CHRISTOFFEL and the covariant derivatives of RICCI. In order not to interrupt the course of the reasoning by which the independence of the equations is shown, the definitions of these new variables and the discussion of such of their properties as are used later have been collected in a preliminary section (§ 2). Since in each case the proof depends on the non-*identical* vanishing of certain determinants, it is found that the work is facilitated by the specialization of the form and of the variables.

The results of the investigation may be expressed in the following

THEOREMS:

I. *The number of parameters of order μ and of the first type (Beltramian invariants) is*

$$(a)\quad \begin{cases} \dfrac{m(m+1)}{2}, & \text{if } \mu = 1, \text{ and } m \leqq n; \\ mn - \dfrac{n(n-1)}{2}, & \text{if } \mu = 1, \text{ and } m \geqq n; \end{cases}$$

$$(b)\quad \begin{cases} \dfrac{mn(n+1)}{2} - \dfrac{(n-m)(n-m-1)}{2}, & \text{if } \mu = 2, \text{ and } m \leqq n; \\ \dfrac{mn(n+1)}{2}, & \text{if } \mu = 2, \text{ and } m \geqq n; \end{cases}$$

$$(c)\quad \frac{mn(n+1)(n+2)}{6} + \frac{n(n-1)}{2}, \qquad \text{if } \mu = 3;$$

$$(d)\quad \frac{m(n+\mu-1)!}{(n-1)!\,\mu!}, \qquad \text{if } \mu \geqq 4;$$

where n = number of variables x, m = number of functions U.

II. *The number of parameters of order μ and of the second type (Minding invariants) is*

$$(a)\quad 0, \qquad \text{if } \mu = 1;$$

* ŻORAWSKI, *Ueber Biegungsinvarianten*, Acta Mathematica, vol. 16 (1892), p. 41.

† LEVI-CIVITA, *Sugli invarianti assoluti*, Atti d. R. Ist. Veneto, ser. 7, vol. 52 (1894), p. 1498.

$$(b)\quad \begin{cases} \dfrac{m(m+3)(m^2-m+2)}{8}, & \text{if } \mu=2, \text{ and } \dfrac{m(m+3)}{2} \leqq n; \\ (n-m)\left\{m+\dfrac{m(m+1)}{2}\right\}-\dfrac{n(n-1)}{2}, & \text{if } \mu=2, \text{ and } \dfrac{m(m+3)}{2} \geqq n; \end{cases}$$

$$(c)\quad \begin{cases} (n-m)\dfrac{m(m+1)(m+2)}{6}+\dfrac{m(m+3)}{2}\left\{n-\dfrac{(m+1)(m+2)}{4}\right\}, \\ \qquad\qquad \text{if } \mu=3, \text{ and } \dfrac{m(m+3)}{2} \leqq n; \\ (n-m)\dfrac{m(m+1)(m+2)}{6}+\dfrac{n(n-1)}{2}, \\ \qquad\qquad \text{if } \mu=3, \text{ and } \dfrac{m(m+3)}{2} \geqq n; \end{cases}$$

$$(d)\quad (n-m)\frac{(m+\mu-1)!}{(m-1)!\,\mu!}, \qquad \text{if } \mu \geqq 4;$$

where n = number of variables x, $n-m$ = number of fixed relations among the variables x.

§ 2. *Differential equations of the problem.*

As in the case of the invariants the group must be "extended" to include the functions involved in the parameters sought.

In dealing with parameters of the second type we may regard $n-m$ of the variables x as functions of the remaining m and use the transformation

$$(1)\qquad Xf \equiv \sum_{r=1}^{n} \xi_r(x_1, \cdots, x_n)\frac{\partial f}{\partial x_r},$$

or we may regard them all as functions of m new independent variables $z_1, \cdots, z_m$; the system of variables $x_1, \cdots, x_n; z_1, \cdots, z_m$, being subjected to the transformation

$$(2)\qquad Yf \equiv \sum_{r=1}^{n} \xi(x_1, \cdots, x_n)\frac{\partial f}{\partial x_r} + \sum_{r=1}^{m} \zeta_r(z_1, \cdots, z_m)\frac{\partial f}{\partial z_r}.$$

The following notation will be used.

$$a_{ik} \equiv \text{coefficient of } dx_i dx_k \text{ in } \phi,$$

$$p_{ik} \equiv \frac{\partial f}{\partial a_{ik}},$$

$$a_{\substack{ik \\ l_1 \cdots l_\mu}} \equiv a_{ikl_1\cdots l_\mu} \equiv \frac{\partial^\mu a_{ik}}{\partial x_{l_1}\cdots\partial x_{l_\mu}},$$

$$p_{\substack{ik \\ l_1 \dots l_\mu}} \equiv p_{ik/l_1 \dots l_\mu} \equiv \frac{\partial f}{\partial a_{ik/l_1 \dots l_\mu}},$$

$$a \equiv |a_{11} \cdots a_{nn}| \equiv \text{discriminant of the form } \phi,$$

$$A_{ik} \equiv \frac{1}{a}\{\text{cofactor of } a_{ik} \text{ in } a\},$$

$$P_{ik} \equiv \frac{\partial f}{\partial A_{ik}},$$

(3) $$\begin{bmatrix} ik \\ l \end{bmatrix} \equiv \frac{1}{2}(a_{\substack{kl \\ i}} + a_{\substack{il \\ k}} - a_{\substack{ik \\ l}}) \equiv$$ CHRISTOFFEL'S three-index symbol of the first kind,

$$q_{\substack{ik \\ l}} \equiv q_{ik/l} \equiv \frac{\partial f}{\partial \begin{bmatrix} ik \\ l \end{bmatrix}},$$

(4) $$\begin{Bmatrix} ik \\ l \end{Bmatrix} \equiv \sum_{\nu=1}^{n} A_{\nu l} \begin{bmatrix} ik \\ \nu \end{bmatrix} \equiv$$ CHRISTOFFEL'S three-index symbol of the second kind,

$$\xi_{\substack{r \\ l_1 \dots l_\mu}} \equiv \xi_{r/l_1 \dots l_\mu} \equiv \frac{\partial^\mu \xi_r}{\partial x_{l_1} \cdots \partial x_{l_\mu}},$$

$$\zeta_{\substack{r \\ l_1 \dots l_\mu}} \equiv \zeta_{r/l_1 \dots l_\mu} \equiv \frac{\partial^\mu \zeta}{\partial z_{l_1} \cdots \partial z_{l_\mu}},$$

$$U_{\substack{h \\ l_1 \dots l_\mu}} \equiv U_{h/l_1 \dots l_\mu} \equiv \frac{\partial^\mu U_h}{\partial x_{l_1} \cdots \partial x_{l_\mu}},$$

$$Q_{\substack{h \\ l_1 \dots l_\mu}} \equiv Q_{h/l_1 \dots l_\mu} \equiv \frac{\partial f}{\partial U_{h/l_1 \dots l_\mu}},$$

(5) $$V_{\substack{h \\ ik}} \equiv V_{h/ik} \equiv U_{\substack{h \\ ik}} - \sum_{s=1}^{n} \begin{Bmatrix} ik \\ s \end{Bmatrix} U_{\substack{h \\ s}} \equiv$$ RICCI'S "second covariant derivative,"

$$R_{\substack{h \\ ik}} \equiv R_{h/ik} \equiv \frac{\partial f}{\partial V_{\substack{h \\ ik}}},$$

(6) $$W_{hh} \equiv \Delta U_h \equiv \sum_{i=1}^{n}\sum_{k=1}^{n} A_{ik} U_{\substack{h \\ i}} U_{\substack{h \\ k}} \equiv$$ BELTRAMI'S first differential parameter,

(7) $$W_{hl} \equiv \nabla U_h U_l \equiv \sum_{i=1}^{n}\sum_{k=1}^{n} A_{ik} U_{\substack{h \\ i}} U_{\substack{l \\ k}} \equiv$$ BELTRAMI'S mixed differential parameter,

$$T_{hl} \equiv \frac{\partial f}{\partial W_{hl}},$$

$$x_{\substack{i\\ l_1 \dots l_\mu}} \equiv x_{i l_1 \dots l_\mu} \equiv \frac{\partial^\mu x_i}{\partial z_{l_1} \cdots \partial z_{l_\mu}},$$

$$\rho_{\substack{i\\ l_1 \dots l_\mu}} \equiv \rho_{i l_1 \dots l_\mu} \equiv \frac{\partial f}{\partial x_{i l_1 \dots l_\mu}},$$

$$(8) \qquad y_{\substack{l\\ st}} \equiv y_{l st} \equiv x_{\substack{l\\ st}} + \sum_{i=1}^{n}\sum_{k=1}^{n} \left\{ \begin{matrix} ik \\ l \end{matrix} \right\} x_{i s} x_{k t},$$

$$\sigma_{\substack{l\\ st}} \equiv \sigma_{l st} \equiv \frac{\partial f}{\partial (y_{l st})},$$

$$(9) \qquad h_{ik} \equiv \sum_{r=1}^{n}\sum_{s=1}^{n} a_{rs} x_{r i} x_{s k}, \qquad \gamma_{ik} = \frac{\partial f}{\partial h_{ik}},$$

$$(10) \qquad h_{i,\,kl} \equiv \sum_{r=1}^{n}\sum_{s=1}^{n} a_{rs} \underset{i}{x_r} \underset{kl}{y_s}, \qquad \gamma_{i,\,kl} = \frac{\partial f}{\partial h_{i,\,kl}},$$

$$(11) \qquad h_{ik,\,lj} \equiv \sum_{r=1}^{n}\sum_{s=1}^{n} a_{rs} \underset{ik}{y_r} \underset{lj}{y_s}, \qquad \gamma_{ik,\,lj} = \frac{\partial f}{\partial h_{ik,\,lj}}.$$

We define symbols H_{ik} by the equations,

$$(12) \qquad \sum_{i=1}^{m} h_{il} H_{ik} = \epsilon_{kl} \qquad (k, l = 1, 2, \cdots m;\ \epsilon_{kl} = 0 \text{ if } k \neq l;\ \epsilon_{kk} = 1),$$

and put

$$(13) \qquad \Lambda_{ik,\,lj} \equiv h_{ik,\,lj} - \sum_{p=1}^{m}\sum_{q=1}^{m} H_{pq} h_{p,\,ik} h_{q,\,lj},$$

$$\lambda_{ik,\,lj} \equiv \frac{\partial f}{\partial \Lambda_{ik,\,lj}}.$$

$$(14) \qquad Xa_{ik} = -\sum_{r=1}^{n} (a_{rk} \underset{i}{\xi_r} + a_{ir} \underset{k}{\xi_r}).^*$$

The quantities A_{ik} satisfy the relations

$$(15) \qquad \sum_{i=1}^{n} a_{il} A_{ik} = \epsilon_{kl},$$

from which we obtain by means of (14)

$$(16) \qquad XA_{ik} = +\sum_{r=1}^{n} (A_{rk} \underset{i}{\xi_r} + A_{ir} \underset{k}{\xi_r}).$$

$$(17) \qquad \underset{l}{Xa_{ik}} = -\sum_{r=1}^{n} (\underset{l}{a_{rk}} \underset{i}{\xi_r} + \underset{l}{a_{ir}} \underset{k}{\xi_r} + \underset{r}{a_{ik}} \underset{l}{\xi_r} + a_{rk} \underset{il}{\xi_r} + a_{ir} \underset{kl}{\xi_r}).^*$$

* Cf. *Invariants*, I, p. 75.

From this and the relation defining $\left[\begin{smallmatrix} ik \\ l \end{smallmatrix}\right]$ we have

$$(18)\qquad X\left[\begin{matrix} ik \\ l \end{matrix}\right] = -\sum_{r=1}^{n}\left(\left[\begin{matrix} rk \\ l \end{matrix}\right]\xi_{ir} + \left[\begin{matrix} ir \\ l \end{matrix}\right]\xi_{kr} + \left[\begin{matrix} ik \\ r \end{matrix}\right]\xi_{lr} + a_{lr}\xi_{ikr}\right),$$

and

$$(19)\qquad X\left\{\begin{matrix} ik \\ l \end{matrix}\right\} = -\sum_{r=1}^{n}\left(\left\{\begin{matrix} rk \\ l \end{matrix}\right\}\xi_{ir} + \left\{\begin{matrix} ir \\ l \end{matrix}\right\}\xi_{kr} - \left\{\begin{matrix} ik \\ r \end{matrix}\right\}\xi_{lr}\right) - \xi_{ikl}.$$

Since

$$(20)\qquad X(U_{l_1 \dots l_\mu l_{\mu+1}}) = \frac{\partial(XU_{l_1 \dots l_\mu})}{\partial x_{l_{\mu+1}}} - \sum_{r=1}^{n}\frac{\partial(U_{l_1 \dots l_\mu})}{\partial x_r}\xi_{rl_{\mu+1}},^{*}$$

we have from

$$(21)\qquad XU_h = 0,$$

$$(22)\qquad XU_{hi} = -\sum_{r=1}^{n}U_{hr}\xi_{ri},$$

and

$$(23)\qquad XU_{hik} = -\sum_{r=1}^{n}\left\{U_{hrk}\xi_{ri} + U_{hir}\xi_{rk} + U_{hr}\xi_{rik}\right\}.$$

From (5) and (23) we have

$$(24)\qquad XV_{hik} = -\sum_{r=1}^{n}\left\{V_{hrk}\xi_{ri} + V_{hir}\xi_{rk}\right\}.$$

From (7), (16), and (22) follows

$$(25)\qquad XW_{hl} = 0.$$

From (2) and (20) we have

$$(26)\qquad Y(x_{ik}) = \sum_{r=1}^{n}x_{rk}\xi_{ir} - \sum_{r=1}^{m}x_{ir}\zeta_{rk},$$

whence

$$(27)\qquad Y(x_{ikl}) = \sum_{r=1}^{n}\sum_{s=1}^{n}x_{rk}x_{sl}\xi_{irs} + \sum_{r=1}^{n}x_{rkl}\xi_{ir} - \sum_{r=1}^{m}(x_{irl}\zeta_{rk} + x_{ikr}\zeta_{rl}) - \sum_{r=1}^{m}x_{ir}\zeta_{rkl}.$$

From (8), (19), (26) and (27) we have

$$(28)\qquad Y(y_{ikl}) = \sum_{r=1}^{n}y_{rkl}\xi_{ir} - \sum_{r=1}^{m}([y_{irl}\zeta_{rk} + y_{irk}\zeta_{rl}] + x_{ikl}\zeta_{rl}).$$

From (14) and (26) we have

$$(29)\qquad Y(h_{ik}) = -\sum_{r=1}^{m}(h_{rk}\zeta_{ir} + h_{ir}\zeta_{rk}).$$

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*LIE-ENGEL, *Transformationsgruppen*, vol. 1, p. 545.

Similarly we obtain

$$(30)\quad Y(h_{i,kl}) = -\sum_{r=1}^{m}(h_{r,kl}\underset{i}{\zeta_r} + h_{i,rl}\underset{k}{\zeta_r} + h_{i,kr}\underset{l}{\zeta_r}) - \sum_{r=1}^{m} h_{ir}\underset{kl}{\zeta_r},$$

and

$$(31)\quad Y(h_{ik,lj}) = -\sum_{r=1}^{m}(h_{rk,lj}\underset{i}{\zeta_r} + h_{ir,lj}\underset{k}{\zeta_r} + h_{ik,rj}\underset{l}{\zeta_r} + h_{ik,lr}\underset{j}{\zeta_r}) - \sum_{r=1}^{m}(h_{r,lj}\underset{ik}{\zeta_r} + h_{r,ik}\underset{lj}{\zeta_r}).$$

In the same way that (16) is derived from (14) and (15), we have from (12) and (29)

$$(32)\quad Y(H_{ik}) = +\sum_{r=1}^{m}(H_{rk}\underset{r}{\zeta_i} + H_{ir}\underset{r}{\zeta_k}).$$

From (13), (30), (31), and (32) we have

$$(33)\quad Y(\Lambda_{ik,lj}) = -\sum_{r=1}^{m}(\underset{i}{\zeta_r}\Lambda_{rk,lj} + \underset{k}{\zeta_r}\Lambda_{ir,lj} + \underset{l}{\zeta_r}\Lambda_{ik,rj} + \underset{j}{\zeta_r}\Lambda_{ik,lr}).$$

Two special cases of this relation are important:

$$(34)\quad Y(\Lambda_{il,il}) = -4\sum_{r=1}^{m}\underset{i}{\zeta_r}\Lambda_{rl,il},$$

$$(35)\quad Y(\Lambda_{ik,ik}) = -2\sum_{r=1}^{m}(\underset{i}{\zeta_r}\Lambda_{rk,ik} + \underset{k}{\zeta_r}\Lambda_{ir,ik}).$$

The differential equations for the parameters of the first type and μth order are obtained by equating to zero the coefficients of the various derivatives of the ξ's in the expression

$$(36)\quad \sum_{i=1}^{n}\sum_{k=1}^{i}\left\{p_{ik}Xa_{ik} + \sum_{l_1=1}^{n}\underset{l_1}{p_{ik}}Xa_{ik}_{l_1} + \cdots + \sum_{l_1=1}^{n}\sum_{l_2=1}^{l_1}\cdots\sum_{l_{\mu-1}=1}^{l_{\mu-2}}\underset{l_1\ldots l_{\mu-1}}{p_{ik}}\underset{l_1\ldots l_{\mu-1}}{Xa_{ik}}\right\}$$
$$+\sum_{h=1}^{m}\left\{\sum_{l_1=1}^{n}\underset{l_1}{Q_h}X(\underset{l_1}{U_h}) + \cdots + \sum_{l_1=1}^{n}\sum_{l_2=1}^{l_1}\cdots\sum_{l_\mu=1}^{l_{\mu-1}}\underset{l_1\ldots l_\mu}{Q_h}\ \underset{l_1\ldots l_\mu}{X(U_h)}\right\}.$$

The equations for the parameters of the second type are similarly formed, save that the derivatives of the x's replace those of the U's, and, if we use the variables z, the coefficients of the derivatives of the ζ's as well as those of the ξ's must be considered.

The variables $a_{ik/l}$ can in these equations be replaced by the three index symbols $\left[\begin{smallmatrix} ik \\ l \end{smallmatrix}\right]$,* and the equations (5) and (8) of the present section show that the variables $U_{h/ik}$ and $x_{l/st}$ can be replaced by $V_{h/ik}$ and $y_{l/st}$ respectively.

The variables a_{ik} can be replaced by the A_{ik}, and in considering parameters of the second type it is convenient to make this substitution.

* Cf. *Invariants*, I, p. 78.

The equations for parameters of the first type of order unity, and of orders not exceeding two, may be derived from the expressions

$$\sum_{i=1}^{n}\sum_{k=1}^{i} p_{ik} X(a_{ik}) + \sum_{h=1}^{m}\sum_{i=1}^{n} Q_{\substack{h \\ i}} X(U_{\substack{h \\ i}}), \tag{37}$$

and

$$\sum_{i=1}^{n}\sum_{k=1}^{i} p_{ik} X(a_{ik}) + \sum_{i=1}^{n}\sum_{k=1}^{i}\sum_{l=1}^{n} q_{\substack{ik \\ l}} X\begin{bmatrix} ik \\ l \end{bmatrix} + \sum_{h=1}^{m}\sum_{i=1}^{n} Q_{\substack{h \\ i}} X(U_{\substack{h \\ i}}) + \sum_{h=1}^{m}\sum_{i=1}^{n}\sum_{k=1}^{i} R_{\substack{h \\ ik}} X(V_{\substack{h \\ ik}}), \tag{38}$$

respectively.

The equations for the parameters of the second type of order unity, and of orders not exceeding two, may be determined from

$$\sum_{i=1}^{n}\sum_{k=1}^{i} P_{ik} Y(A_{ik}) + \sum_{h=1}^{m}\sum_{i=1}^{n} \rho_{\substack{i \\ h}} Y(x_{\substack{i \\ h}}), \tag{39}$$

and

$$\sum_{i=1}^{n}\sum_{k=1}^{i} P_{ik} Y(A_{ik}) + \sum_{i=1}^{n}\sum_{k=1}^{i}\sum_{l=1}^{n} q_{\substack{ik \\ l}} Y\begin{bmatrix} ik \\ l \end{bmatrix} + \sum_{h=1}^{m}\sum_{i=1}^{n} \rho_{\substack{i \\ h}} Y(x_{\substack{i \\ h}}) + \sum_{h=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{i} \sigma_{\substack{h \\ ik}} Y(y_{\substack{h \\ ik}}), \tag{40}$$

respectively.

We shall find it convenient to denote any equation by the $\xi_{r/i}$, $\xi_{r/ik}$, $\zeta_{r/i}$ or $\zeta_{r/ik}$ of which its left-hand member is, in the above expressions, the coefficient.

If we proceed to form the equations, we notice that the introduction of the variables $V_{h/ik}$ and $y_{h/ik}$ has removed the second derivatives of the ξ's from the expressions (38) and (40), save as they appear in the expressions $X\left[\begin{smallmatrix} ik \\ l \end{smallmatrix}\right]$.

The corresponding equations are, then,

$$(\xi_{\substack{r \\ ik}}) \qquad -\sum_{l=1}^{n} a_{lr} q_{\substack{ik \\ l}} = 0 \quad (r, i = 1, 2, \cdots, n;\ k = 1, 2, \cdots, i). \tag{41}$$

But these are equivalent to *

$$q_{\substack{ik \\ l}} = 0 \qquad (i, l = 1, 2, \cdots, n;\ k = 1, 2, \cdots, i). \tag{42}$$

Hence in both cases the $q_{ik/l}$ and the corresponding equations $\xi_{r/ik}$ may be dismissed from consideration.

The equations then take the form:

I_1. Type I, order 1,

$$(\xi_{\substack{r \\ i}}) \qquad -\sum_{k=1}^{n}(1+\epsilon_{ik}) a_{rk} p_{ik} - \sum_{h=1}^{m} U_{\substack{h \\ r}} Q_{\substack{h \\ i}} = 0 \quad \begin{pmatrix} r, i = 1, 2, \cdots n \\ \epsilon_{ik} = 0,\ i \neq k \\ \epsilon_{ii} = 1 \end{pmatrix}.$$

* Cf. *Invariants*, I, p. 78.

I_2. Type I, orders 1 and 2,

$$(\xi_r^i) \qquad -\sum_{k=1}^{n}(1+\epsilon_{ik})a_{rk}p_{ik} - \sum_{h=1}^{m}\sum_{k=1}^{n}(1+\epsilon_{ik})V_{rk}^{h}R_{ik}^{h} - \sum_{h=1}^{m}U_r^h Q_i^h = 0$$

$$(r, i = 1, 2, \cdots, n).$$

II_1. Type II, order 1,

$$(\xi_r^i) \qquad \sum_{k=1}^{n}(1+\epsilon_{ik})A_{rk}P_{ik} + \sum_{k=1}^{m}x_r^k\rho_i^k = 0 \qquad (r, i = 1, 2, \cdots, n),$$

$$(\zeta_i^r) \qquad -\sum_{k=1}^{n}x_k^r\rho_k^i = 0 \qquad (r, i = 1, 2, \cdots, m).$$

II_2. Type II, orders 1 and 2,

$$(\xi_r^i) \qquad \sum_{k=1}^{n}(1+\epsilon_{ik})A_{rk}P_{ik} + \sum_{k=1}^{m}x_k^r\rho_k^i + \sum_{k=1}^{m}\sum_{l=1}^{k}y_{kl}^r\sigma_{kl}^i = 0 \quad (r, i = 1, 2, \cdots, n),$$

$$(\zeta_{ik}^r) \qquad -\left\{\sum_{k=1}^{n}x_k^r\rho_k^i + \sum_{k=1}^{n}\sum_{l=1}^{m}(1+\epsilon_{il})y_{kl}^r\sigma_{il}^k\right\} = 0 \qquad (r, i = 1, 2, \cdots, m).$$

$$(\zeta_{ik}^r) \qquad -\sum_{l=1}^{n}x_l^r\sigma_{ik}^l = 0 \qquad (r, i, k = 1, 2, \cdots, m;\ k \leqq i)$$

The explicit form of the equations for parameters of higher order is, for our present purpose, unessential.

§ 3. *Parameters of the first type and first order.*

THEOREM. *If $m \leqq n$ there are in general*

$$\frac{m(m+1)}{2}$$

parameters of the first type and first order.

If $m \geqq n$ there are

$$\frac{n(n+1)}{2} + n(m-n) = nm - \frac{n(n-1)}{2}$$

such parameters.

They may be taken as the Beltramian parameters

$$W_{hh} \equiv \Delta U_h \equiv \sum_{i=1}^{n}\sum_{k=1}^{n}A_{ik}U_i^h U_k^h,$$

$$W_{hl} \equiv \nabla U_h U_l \equiv \sum_{i=1}^{n}\sum_{k=1}^{n}A_{ik}U_i^h U_k^l.$$

The equations for the parameters are

$$(\underset{i}{\xi}_r) \qquad -\sum_{k=1}^{n}(1+\epsilon_{ik})a_{rk}p_{ik}-\sum_{h=1}^{m}U_{\underset{r}{h}}\,Q_{\underset{i}{h}}=0 \quad (r, i=1, 2, \cdots, n).$$

We will suppose the equations arranged in the following order,

$$\underset{1}{\xi}_1, \underset{1}{\xi}_2, \cdots, \underset{1}{\xi}_n; \ \underset{2}{\xi}_1, \underset{2}{\xi}_2, \cdots, \underset{2}{\xi}_n; \ \cdots; \ \underset{n}{\xi}_1, \underset{n}{\xi}_2, \cdots, \underset{n}{\xi}_n;$$

and the variables p_{ik} and $Q_{h/i}$ in the order,

$$p_{11}, p_{12}, \cdots, p_{1n}; \ p_{22}, p_{23}, \cdots, p_{2n}; \ \cdots; \ p_{n-1\,n-1}, p_{n-1\,n}; \ p_{n\,n};$$

$$\underset{1}{Q}_1, \underset{2}{Q}_1, \cdots, \underset{n}{Q}_1; \ \underset{1}{Q}_2, \underset{2}{Q}_2, \cdots, \underset{n}{Q}_2; \ \cdots; \ \underset{1}{Q}_m, \underset{2}{Q}_m, \cdots, \underset{n}{Q}_m.$$

The rows of the matrix of the coefficients of these equations will be denoted by the symbols $\xi_{r/i}$ used to represent the corresponding equations. The columns of the matrix will be denoted by the symbols p_{ik} of which the corresponding elements are coefficients.

We suppose the variables so numbered that the determinant of order $n(n+1)/2$ formed from the rows

$$\underset{1}{\xi}_1, \underset{1}{\xi}_2, \cdots, \underset{1}{\xi}_n; \ \underset{2}{\xi}_2, \underset{2}{\xi}_3, \cdots, \underset{2}{\xi}_n; \ \cdots; \ \underset{n-1}{\xi}_{n-1}, \underset{n-1}{\xi}_n; \ \underset{n}{\xi}_n,$$

and the columns

$$p_{11}, p_{12}, \cdots, p_{1n}; \ p_{22}, p_{23}, \cdots, p_{2n}; \ p_{33}, p_{34}, \cdots, p_{3n}; \ \cdots; \ p_{n-1\,n-1}, p_{n-1\,n}; p_{n\,n},$$

does not vanish. This is always possible. *

Suppose now $m=1$, i. e., that a single function U_1 is considered. Build the determinants whose rows are, in order

$$\underset{1}{\xi}_1, \underset{1}{\xi}_2, \cdots, \underset{1}{\xi}_n; \ \underset{2}{\xi}_1, \underset{2}{\xi}_2, \cdots, \underset{2}{\xi}_n; \ \underset{3}{\xi}_2, \underset{3}{\xi}_3, \cdots, \underset{3}{\xi}_n; \ \underset{4}{\xi}_3, \underset{4}{\xi}_4 \cdots, \underset{4}{\xi}_n; \cdots; \underset{n}{\xi}_{n-1}, \underset{n}{\xi}_n;$$

and whose columns are, in order

$$p_{11}p_{12}\cdots p_{1n}; \ \underset{2}{Q}_1, p_{22}, \cdots, p_{2n}; \ \underset{3}{Q}_1, p_{33}, \cdots, p_{3n}; \ \underset{4}{Q}_1, p_{44}, \cdots, p_{4n}; \cdots; \ \underset{n}{Q}_1, p_{n\,n}.$$

This determinant has the peculiarity that coaxial with its principal diagonal lie minors of order $n, n, n-1, n-2, \cdots, 2$, respectively, and that at the right of these minors and above them the elements are all zeros. Hence the determinant consists of the product of these minors. These minors are n in number. The first is the discriminant a of the form. The remaining $n-1$ are all polynomials in the derivatives $U_{1/k}$, among the coefficients of each one of which there is at least one which does not vanish. Hence there can be assigned values of $U_{11}\cdots U_{1n}$

* Cf. *Invariants*, I, p. 88.

such that *no one* of these minors vanishes. Hence none of them vanishes identically. Consequently the determinant we have just formed does not vanish identically.

Take now $m = 2$, and form the determinant, whose rows, in order, are

$$\xi_{1\,1}, \xi_{2\,1}, \cdots, \xi_{n\,1}; \xi_{1\,2}, \xi_{2\,2}, \cdots, \xi_{n\,2}; \xi_{1\,3}, \xi_{2\,3}, \xi_{3\,3}, \cdots, \xi_{n\,3}; \xi_{2\,4}, \xi_{3\,4}, \xi_{4\,4}, \cdots, \xi_{n\,4}; \cdots; \xi_{n-2\,n}, \xi_{n-1\,n}, \xi_{n\,n};$$

and whose columns, in order, are

$$p_{11}, p_{12}, \cdots, p_{1n}; Q_{1\,2}, p_{22}, \cdots, p_{2n}; Q_{2\,3}, Q_{1\,3}, p_{33}, \cdots, p_{3n}; Q_{2\,4}, Q_{1\,4}, p_{44}, \cdots, p_{4n}; \cdots; Q_{2\,n}, Q_{1\,n}, p_{n\,n}.$$

To this determinant the same reasoning applies as to the one before considered. It does not, therefore, identically vanish.

The process above indicated can be continued in the same way for $m = 3, 4,$ etc. For $m = 1$ the order of the determinant formed is

$$n + n + (n-1) + (n-2) + \cdots + 3 + 2 = \frac{n(n+1)}{2} + n - 1.$$

For $m = 2$ the order is

$$n + n + n + (n-1) + \cdots + 4 + 3 = \frac{n(n+1)}{2} + 2n - \frac{3 \cdot 2}{2},$$

and in general the order is

$$\frac{n(n+1)}{2} + mn - \frac{m(m+1)}{2}.$$

This process can, however, be continued no further than $m = n - 1$, for then

$$\frac{n(n+1)}{2} + mn - \frac{m(m+1)}{2} = n^2,$$

and all the rows of the matrix have been used. We have therefore proved that if $m \leqq n-1$ functions U are considered, the matrix of the equations for the parameters of the first type and order contains at least one non-vanishing determinant of order

$$\frac{n(n+1)}{2} + mn - \frac{m(m+1)}{2},$$

and if $m \geqq n-1$ it contains at least one non-vanishing determinant of order n^2. Hence

If $m \leqq n-1$ *there are at least*

$$\frac{n(n+1)}{2} + mn - \frac{m(m+1)}{2}$$

independent equations in the system which determines the parameters of first order and type. If $m \geqq n-1$ *all the equations of this system are independent.*

It remains now to show that if $m \leqq n-1$ there are *not more than*

$$\frac{n(n+1)}{2} + mn - \frac{m(m+1)}{2}$$

independent equations in the system. To do this we make use of the fact that certain solutions of the equations, namely, BELTRAMI's first differential parameters,

$$\Delta U_h \equiv \sum_{i=1}^{n}\sum_{k=1}^{n} A_{ik} U_{\underset{i}{h}} U_{\underset{k}{h}} \equiv W_{hh},$$

and mixed differential parameters,

$$\nabla U_h U_l \equiv \sum_{i=1}^{n}\sum_{k=1}^{n} A_{ik} U_{\underset{i}{h}} U_{\underset{k}{l}} \equiv W_{hl},$$

are known. Moreover these solutions regarded as functions of the explicitly expressed variables A_{ik}, $U_{h/l}$ are functionally independent as may be seen from the fact that if

$$U_r = x_1 + x_2 + \cdots + x_r \qquad (r \leqq m),$$

the Jacobian

$$\frac{\partial(W_{11}W_{21}W_{22}W_{31}W_{32}W_{33}\cdots W_{mm})}{\partial(A_{11}A_{21}A_{22}A_{31}A_{32}A_{33}\cdots A_{mm})}$$

has the value unity, and does not, therefore, vanish. If $m \geqq n$ the same set of values for $U_1 \cdots U_n$ shows that the Jacobian

$$\frac{\partial(W_{11}W_{21}W_{22}\cdots W_{nn}, W_{1\,n+1}, W_{2\,n+1}\cdots W_{n\,n+1}\cdots W_{1m}, \cdots, W_{nm})}{\partial(A_{11}A_{21}A_{22}\cdots A_{nn}, U_{\underset{1}{n+1}}, U_{\underset{2}{n+1}}\cdots U_{\underset{n}{n+1}}\cdots U_{\underset{1}{m}}\cdots U_{\underset{n}{m}})}$$

does not vanish.

We have seen that there are at least $n(n+1)/2 + mn - m(m+1)/2$ independent equations if $m \leqq n-1$ and that all the equations are independent if $m \geqq n-1$.

Hence there cannot be more than

$$\frac{n(n+1)}{2} + mn - \left\{\frac{n(n+1)}{2} + mn - \frac{m(m+1)}{2}\right\} = \frac{m(m+1)}{2}$$

independent parameters if $m \leqq n-1$, and not more than

$$\frac{n(n+1)}{2} + mn - n^2 = \frac{n(n+1)}{2} + n(m-n)$$

if $m \geqq n-1$. But we have actually formed this number of independent parameters. The theorem is therefore proved.

§4. *Parameters of the first type and second order.*

THEOREM: *The number of parameters of the first type and second order is, in general,*

$$\frac{mn(m+1)}{2} - \frac{(n-m)(n-m-1)}{2}, \qquad \text{if } m \leqq n,$$

and

$$\frac{mn(n+1)}{2}, \qquad \text{if } m \geqq n.$$

The equations determining the parameters of order not exceeding two may be written:

$$(\xi_r) \qquad -\left\{ \sum_{k=1}^{n} (1+\epsilon_{ik}) a_{rk} p_{ik} + \sum_{h=1}^{m} \sum_{k=1}^{n} (1+\epsilon_{ik}) V_{rk}^{h} R_{ik}^{h} + \sum_{h=1}^{m} U_{r}^{h} Q_{i}^{h} \right\} = 0$$

$$(r, i = 1, 2, \cdots, n).$$

If we omit the terms

$$\sum_{h=1}^{m} U_{r}^{h} Q_{i}^{h},$$

there remain the equations which determine the simultaneous invariants of order zero for the $m+1$ forms

$$\sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} dx_i dx_k,$$

$$\phi_h = \sum_{i=1}^{n} \sum_{k=1}^{n} V_{ik}^{h} dx_i dx_k \qquad (h = 1, 2, \cdots, m).$$

But these equations are all independent. *

Consequently the equations of the present case, which differ from these only by the addition of terms are also all independent.

The number of equations is n^2, the number of variables,

$$\frac{mn(n+1)}{2} + \frac{n(n+1)}{2} + mn.$$

The number of parameters of orders one and two is, therefore,

$$\frac{mn(n+1)}{2} + \frac{n(n+1)}{2} + mn - n^2 = \frac{mn(n+3)}{2} - \frac{n(n-1)}{2}.$$

The number of parameters of the first order is

$$\frac{m(m+1)}{2}, \qquad \text{if } m \leqq n-1,$$

and

$$\frac{n(n+1)}{2} + mn - n^2, \qquad \text{if } m \geqq n-1.$$

* This theorem is involved in the theorem concerning simultaneous invariants, (*Invariants*, I, p. 91). The proof follows the same lines as that used in *Invariants* I, § 5, for a similar purpose.

Hence the number of parameters of the second order is

$$\frac{mn(n+1)}{2} - \frac{(n-m)(n-m-1)}{2}, \qquad \text{if } m \leqq n-1;$$

and

$$\frac{mn(n+1)}{2}, \qquad \text{if } m \geqq n-1.^*$$

Q. E. D.

§ 5. *Parameters of the second type and first order.*

THEOREM. *There are in general no parameters of the second type and first order.*

The equations for the parameters of the second type and first order may be written:

$$(\xi_i) \qquad \sum_{k=1}^{n}(1+\epsilon_{ik})A_{rk}P_{ik} + \sum_{k=1}^{m} x_r \rho_i = 0 \qquad (r, i = 1, 2, \cdots, n),$$

$$(\zeta_r) \qquad -\sum_{k=1}^{n} x_k \rho_k = 0 \qquad (r, i = 1, 2, \cdots, m).$$

We note that the equation $\xi_{i/r}$ of this set involves the indices i and r in the in the same way as does the equation $\xi_{r/i}$ of the set which determines the parameters of the *first* type and order. Consequently we shall expect to find that the functions

$$h_{ik} \equiv \sum_{r=1}^{n}\sum_{s=1}^{n} a_{rs} x_r x_s,$$

are solutions of these equations; and this expectation is confirmed by equations (29) of § 2.

Moreover, the operations by which we proved the independence of equations $\xi_{r/i}$ of § 3 can be repeated step for step here. Hence as $m \leqq n-1$ the equations $\xi_{r/i}$ have precisely $m(m+1)/2$ independent solutions, viz., the functions h_{ik}.

We may therefore introduce these as new variables into the equations $\zeta_{r/i}$. These, then become, however, by means of equation (29), § 2,

$$(\zeta_r) \qquad \sum_{k=1}^{m}(1+\epsilon_{ik})h_{rk}\gamma_{ik} = 0,$$

i. e., the equations which determine the invariants of order zero for a form in m variables. This system consists of m^2 equations in $m(m+1)/2$ variables. Its matrix contains at least one determinant of order $m(m+1)/2$ which does not

* Evidently

$$\frac{mn(n+1)}{2} = \frac{mn(n+1)}{2} - \frac{(n-m)(n-m-1)}{2} \quad \text{when} \quad m=n-1 \quad \text{or} \quad m=n.$$

formally vanish identically.* Now the h_{ik}'s are independent rational integral functions of the $(n+1)/2 + mn > m(m+1)/2$ variables a_{ik}, x_{ik}. Hence these variables can be given such values that this determinant does not vanish. This set of m^2 equations contains therefore in general as many independent equations as variables. Hence it has no solutions.

Hence *there are no parameters of the second type and first order.*

§ 6. *Parameters of the second type and second order.*

THEOREM. *The number of parameters of the second type and second order is, in general,*

$$\frac{m(m+3)(m^2-m+2)}{8}, \quad \text{if} \quad \frac{m(m+3)}{2} \leqq n,$$

and

$$\frac{(n-m)m(m+3)}{2} - n\frac{(n-1)}{2}, \quad \text{if} \quad \frac{m(m+3)}{2} \geqq n,$$

The equations which the parameters of the second order and type must satisfy are:

$$(\xi_{r\ i}) \qquad \sum_{k=1}^{n}(1+\epsilon_{ik})A_{rk}P_{ik} + \sum_{k=1}^{m} x_{r\ k}\rho_{i\ k} + \sum_{k=1}^{m}\sum_{l=1}^{k} y_{r\ kl}\sigma_{i\ kl} = 0 \qquad (r, i = 1, 2, \cdots n),$$

$$(\zeta_{r\ i}) \qquad -\sum_{k=1}^{n} x_{k\ r}\rho_{k\ i} - \sum_{k=1}^{n}\sum_{l=1}^{m}(1+\epsilon_{il})y_{k\ rl}\sigma_{k\ il} = 0 \qquad (r, i = 1, 2, \cdots m),$$

$$(\zeta_{r\ ik}) \qquad -\sum_{l=1}^{n} x_{l\ ik}\sigma_{l\ ik} = 0 \qquad (r, i, k = 1, 2, \cdots m;\ k \leqq i).$$

Equations (29), (30), (31) and (33) of § 2 show that the functions h_{ik}, $h_{i,kl}$, $h_{ik,lj}$ and $\Lambda_{ik,lj}$ † are solutions of the equations ξ_{ir}, and the results of § 3 show that the number of independent solutions of these equations is

$$\frac{m(m+1)(m+2)(m+3)}{8} \quad \text{if} \quad \frac{m(m+3)}{2} \leqq n,$$

and

$$\frac{mn(m+3)}{2} - \frac{n(n-1)}{2}, \quad \text{if} \quad \frac{m(m+3)}{2} \geqq n.$$

We shall first show that these solutions may be selected from the solutions h_{ik}, $h_{i,kl}$, $\Lambda_{ik,lj}$. To this end we arrange these functions in the following order:

* Cf. *Invariants*, I, p. 88.

† The functions $h_{ik,lj}$ are those which naturally present themselves first in the solution. The functions $\Lambda_{ik,lj}$ are introduced to meet the exigencies of the present section.

$$
\begin{array}{l}
h_{11}\ h_{12}\ \cdots h_{1m}\ h_{22}\ \cdots h_{2m}\ \cdots h_{m-1\,m-1}\ h_{m-1\,m}\ h_{mm}\ ;\\
h_{1,11}\ h_{1,12}\ \cdots h_{1,1m}\ h_{1,22}\ \cdots h_{1,2m}\ \cdots h_{1,m-1\,m-1}\ h_{1,m-1\,m}\ h_{1,mm}\ ;\\
h_{2,11}\ h_{2,12}\ \cdots h_{2,1m}\ h_{2,22}\ \cdots h_{2,2m}\ \cdots h_{2,m-1\,m-1}\ h_{2,m-1\,m}\ h_{2,mm}\ ;\\
\cdots\cdots\cdots

$$
\begin{array}{llllllllllll}
x_1 & x_2 & \cdots & x_m & x_2 & \cdots & x_m & \cdots & x_{m-1} & x_m & x_m \\
1 & 1 & & 1 & 2 & & 2 & & m-1 & m-1 & m \\
y_1 & y_1 & \cdots & y_1 & y_1 & \cdots & y_1 & \cdots & y_1 & y_1 & y_1 \\
11 & 12 & & 1m & 22 & & 2m & & m-1\ m-1 & m-1\ m & mm \\
y_2 & y_2 & \cdots & y_2 & y_2 & \cdots & y_2 & \cdots & y_2 & y_2 & y_2 \\
11 & 12 & & 1m & 22 & & 2m & & m-1\ m-1 & m-1\ m & mm \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
y_m & y_m & \cdots & y_m & y_m & \cdots & y_m & \cdots & y_m & y_m & y_m \\
11 & 12 & & 1m & 22 & & 2m & & m-1\ m-1 & m-1\ m & mm \\
y_{m+1} & & & & & & & & & & \\
11 & & & & & & & & & & \\
y_{m+1} & y_{m+2} & & & & & & & & & \\
12 & 12 & & & & & & & & & \\
y_{m+1} & y_{m+2} & & y_{m+3} & & & & & & & \\
13 & 13 & & 13 & & & & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & & & & & & \\
y_{m+1} & y_{m+2} & & y_{m+3} & & \cdots & & & y_n & & \\
1,m-1 & 1,m-1 & & 1,m-1 & & & & & 1,m-1 & & \\
y_{m+1} & y_{m+2} & & y_{m+3} & & \cdots & & & y_n & & \\
1m & 1m & & 1m & & & & & 1m & & \\
y_{m+1} & y_{m+2} & & y_{m+3} & & \cdots & & & y_n & & \\
22 & 22 & & 22 & & & & & 22 & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\
y_{m+1} & y_{m+2} & & y_{m+3} & & \cdots & & & y_n & & \\
mm & mm & & mm & & & & & mm & &
\end{array}
$$

and show that this Jacobian does not identically vanish. To this end we show that, for a particular form ϕ, and a particular set of relations between the x's and z's, J takes a value different from zero.

We will suppose, first, that ϕ has *constant* coefficients and that none of the minors of its discriminant of any order are zero. The relations between the x's and the z's shall be:

$$x_r = \sum_{i=1}^{m} A_{ri} z_i + \tfrac{1}{2} \sum_{i=1}^{m}\sum_{k=1}^{m} \underset{ik}{B_r} z_i z_k \qquad (r = 1, 2, \cdots, n;\ \underset{ik}{B_r} = \underset{ki}{B_r}),$$

where the A_{ri} have the significance we have already attached to these symbols and the $\underset{ik}{B_r}$ are constants which we shall later subject to conditions.

Since ϕ has constant coefficients,

$$\underset{ik}{x_r} = \underset{ik}{y_r}.$$

From the above we have

$$\underset{i}{x_r} = A_{ri} + \sum_{k=1}^{m} \underset{ik}{B_r} z_k,$$

and

$$\underset{ik}{y_r} = \underset{ik}{x_r} = \underset{ik}{B_r},$$

If $z_1 = z_2 \cdots = z_m = 0$, $x_1 = x_2 \cdots = x_n = 0$, and $\underset{i}{x_r} = A_{ri}$.

We shall consider the value of J for this set of values of the variables.

Now

$$\frac{\partial h_{ik}}{\partial x_{\substack{r\\ i}}} = 0, \text{ unless } l = i \text{ or } l = k;$$

$$\frac{\partial h_{ik}}{\partial x_{\substack{r\\ i}}} = \sum_{s=1}^{n} a_{rs} x_{\substack{s\\ k}} \text{ if } k \neq i; \qquad \frac{\partial h_{ii}}{\partial x_{\substack{r\\ i}}} = 2 \sum_{s=1}^{n} a_{rs} x_{\substack{s\\ i}}.$$

But in the special case in question

$$\sum_{s=1}^{n} a_{rs} x_{\substack{s\\ k}} = \sum_{s=1}^{n} a_{rs} A_{sk} = \epsilon_{rk}.$$

Hence

$$\frac{\partial h_{ik}}{\partial x_{\substack{r\\ i}}} = 0, \text{ if } r \neq k: \qquad \frac{\partial h_{ik}}{\partial x_{\substack{k\\ i}}} = 1, \text{ if } i \neq k; \qquad \frac{\partial h_{ii}}{\partial x_{\substack{i\\ i}}} = 2.$$

Now h_{ik} does not contain the variables $y_{r/st}$. Hence J breaks up into the product of two Jacobians, one J_1, that of the h_{ik} with respect to the variables $x_{r/i}$, the other J_2, that of the remaining functions with respect to the remaining variables.

It is easily seen that J_1 has the value 2^m.

Consider now J_2.

$$\frac{\partial h_{i,\,kl}}{\partial y_{\substack{r\\ st}}} = 0 \text{ unless } \left\{ \begin{matrix} s = k \\ t = l \end{matrix} \right\}, \qquad \text{or} \left\{ \begin{matrix} t = k \\ s = l \end{matrix} \right\}, \qquad \frac{\partial h_{i,\,kl}}{\partial y_{\substack{r\\ kl}}} = \sum_{s=1}^{n} a_{rs} x_{\substack{s\\ i}}.$$

In the special case under consideration $\partial h_{i,\,kl}/\partial y_{r/kl} = \epsilon_{ir}$.

Now the index i in $h_{i,\,kl}$ is always $\leqq m$. Hence if $r = m+1, m+2, \cdots$ or n,

$$\frac{\partial h_{i,\,kl}}{\partial y_{\substack{r\\ st}}} = 0.$$

Hence J_2 in turn breaks up into the product of two Jacobians J_2' and J_3; where

$$J_2' = \frac{\partial (h_{1,\,11} h_{1,\,12} \cdots h_{1,\,mm} h_{2,\,11} \cdots h_{2,\,mm} \cdots h_{m,\,mm})}{\partial (y_{1/11} y_{1/12} \cdots y_{1/mm} y_{2/11} \cdots y_{2/mm} \cdots y_{m/mm})}.$$

Therefore

$$J_2'|_{x=0} = 1.$$

The Jacobian J_3 is that of the remaining functions with respect to the remaining variables. We suppose both functions and variables to be arranged in the order given above.

Now,

$$\frac{\partial \Lambda_{ik,\,lj}}{\partial y_{r/st}} = 0, \quad \text{unless} \quad \left\{ \begin{matrix} s = i \\ t = k \end{matrix} \right\}, \left\{ \begin{matrix} s = k \\ t = i \end{matrix} \right\}, \left\{ \begin{matrix} s = l \\ t = j \end{matrix} \right\} \quad \text{or} \quad \left\{ \begin{matrix} s = j \\ t = l \end{matrix} \right\},$$

But if the pairs of indices ik and lj are different,

$$\frac{\partial \Lambda_{ik,\,lj}}{\partial y_{r\,ik}} = \sum_{s=1}^{n} a_{rs} y_{s\,lj} - \sum_{q=1}^{m} h_{q,\,lj} \sum_{p=1}^{m} H_{pq} \sum_{s=1}^{n} a_{sr} x_{s\,q},$$

and if they are the same

$$\frac{\partial \Lambda_{ik,\,ik}}{\partial y_{r\,ik}} = 2 \sum_{s=1}^{n} a_{rs} y_{s\,ik} - 2 \sum_{q=1}^{m} h_{q,\,ik} \sum_{p=1}^{m} H_{pq} \sum_{s=1}^{n} a_{sr} x_{s\,q}.$$

In the present case, since $q \leqq m$ and $r > m$, $q \neq r$, and

$$\sum_{s=1}^{n} a_{sr} x_{s\,q} = \sum_{s=1}^{n} a_{sr} A_{sq} = \epsilon_{qr} = 0.$$

Hence

$$\frac{\partial \Lambda_{ik,\,lj}}{\partial y_{r\,ik}} = \sum_{s=1}^{n} a_{rs} B_{s\,lj}, \qquad \frac{\partial \Lambda_{ik,\,ik}}{\partial y_{r\,ik}} = 2 \sum_{s=1}^{n} a_{rs} B_{s\,ik}.$$

From these equations we see that the only elements of J_3 arising from $\Lambda_{ik,\,lj}$ lie in the rows containing derivatives with respect to $y_{r\,ik}$ and $y_{r\,lj}$. Moreover in the row $y_{r\,ik}$ the indices of the B_s coming from $\Lambda_{ik,\,lj}$ are l, j; and in the row $y_{r\,lj}$ they are i, k. If, bearing this in mind, we consider the order in which both functions and variables are arranged, we see that the determinant J_3 contains a series of minors arranged in order with their principal diagonals lying along that of ω_3, and that below and at the left of these minors all the elements of J_3 are zero. Hence J_3 is the product of these minors. The first $n - m$ of these minors are determinants of orders $1, 2, \cdots, n - m$ respectively. The remainder are of order $n - m$.

Consider a typical one of these minors, e. g., the one whose columns are $1k, 1k;\ 1k-1, 1k;\ 1k-2, 1k;\ \cdots 1k-(n-m-1), 1k$; and whose rows are $m+1_{1k}, m+2_{1k}, m+3_{1k}, \cdots n_{1k}$. It is

	$1k, 1k$	$1k-1, 1k$	$1k-2, 1k$	$\cdots 1k-(n-m-1), 1k$
$m+1_{1k}$	$2\sum_{s=1}^{n} a_{m+1\,s} B_{s\,1k}$	$\sum_{s=1}^{n} a_{m+1\,s} B_{s\,1k-1}$	$\sum_{s=1}^{n} a_{m+1\,s} B_{s\,1k-2}$	$\cdots \sum_{s=1}^{n} a_{m+1\,s} B_{s\,1k-n+m+1}$
$m+2_{1k}$	$2\sum_{s=1}^{n} a_{m+2\,s} B_{s\,1k}$	$\sum a_{m+2\,s} B_{s\,1k-1}$	$\sum_{s=1}^{n} a_{m+2\,s} B_{s\,1k-2}$	$\cdots \sum_{s=1}^{n} a_{m+2\,s} B_{s\,1k-n+m+1}$
$m+3_{1k}$	$2\sum_{s=1}^{n} a_{m+3\,s} B_{s\,1k}$	$\sum a_{m+3\,s} B_{s\,1k-1}$	$\sum_{s=1}^{n} a_{m+3\,s} B_{s\,1k-2}$	$\cdots \sum_{s=1}^{n} a_{m+3\,s} B_{s\,1k-n+m+1}$
	. . .	. . .	. . .	. . .
n_{1k}	$2\sum_{s=1}^{n} a_{ns} B_{s\,1k}$	$\sum a_{ns} B_{s\,1k-1}$	$\sum_{s=1}^{n} a_{ns} B_{s\,1k-2}$	$\cdots \sum_{s=1}^{n} a_{ns} B_{s\,1k-n+m+1}$

The arrangement of the functions Λ is such that the first column of this determinant contains B's which have not before appeared. The remaining columns have however all appeared. We will suppose that not all the determinants of order $n-m-1$ formed from these columns vanish. Then because of the non-vanishing of the minors of the discriminant of ϕ it is possible to choose the quantities $B_{1/1k}B_{2/1k}\cdots B_{n/1k}$ in such a way that the elements in the first column of the determinant in question shall have any values whatever. Hence they may be so chosen that the determinant does not vanish.

Beginning now with the minor of order unity in the column 11, 11 and row $m+1$, we may choose the quantities $B_{s/11}$ so that it does not vanish. Passing to the minor of order two in the next two columns and rows, we may choose the $B_{s/12}$ so that it does not vanish and so on.

It is therefore possible to choose the B's so that J_3 does not vanish. We have found that with our choice of the A's, J_1 and J_2' did not vanish. Hence for this form ϕ and these relations between the x's and z's J does not vanish for the values $z_1=z_2\cdots=z_m=0$.

Hence J does not vanish identically, and therefore the functions chosen are *independent* solutions of the equations $\xi_{i/r}$.

In the case $m(m+3)/2\leqq n$, the solutions are all independent and the foregoing work is unnecessary.

Introduce now these solutions as new variables into the equations $\zeta_{r/i}$ and $\zeta_{r/ik}$. The latter equations become

$$\sum_{i=1}^{m} h_{ir}\gamma_{i,\,kl}=0 \qquad (r,\,k,\,l=1,\,2,\,\cdots,\,m\,;\,l\leqq k).$$

Now the determinant $h\equiv|h_{11}\cdots h_{mm}|$ is easily seen to have the value $|A_{11}\cdots A_{mm}|$ in the case we are considering, and because of the non-vanishing of the minors of the discriminant of ϕ it follows that in this case

$$h\neq 0.$$

Hence *

$$\gamma_{i,\,kl}=0.$$

Consequently we may dismiss the functions $h_{i,\,kl}$ and the equations $\zeta_{i/kl}$ from further consideration.

There remain only the equations $\zeta_{r/i}$ to be considered. They are m^2 in number. The number of variables is the number of independent solutions of the equations $\xi_{i/r}$ diminished by $m^2(m+1)/2$, the number of functions $h_{i,\,kl}$.

The excess of variables over equations is therefore

$$\frac{m(m+1)(m+2)(m+3)}{8}-\frac{m^2(m+1)}{2}-m^2=\frac{m(m+3)}{8}(m^2-m+2),$$

$$\text{if } \frac{m(m+3)}{2}\leqq n;$$

* Cf. *Invariants*, I, p. 78.

and

$$\phi(n, m) \equiv (n-m)\frac{m(m+3)}{2} - \frac{n(n-1)}{2}, \qquad \text{if } \frac{m(m+3)}{2} \geqq n.$$

It is important to note some properties of the function $\phi(n, m)$.

$$\frac{\partial \phi(n, m)}{\partial n} = \frac{m(m+3)}{2} - (n - \tfrac{1}{2}).$$

Hence if $m(m+3)/2 \geqq n$, in which case only is $\phi(n, m)$ used, $\partial\phi/\partial n > 0$. But the least value of n is $m+1$, and $\phi(m+1, m) = m$. Hence if $m+1 \leqq n \leqq m(m+3)/2$, then

$$\phi(n, m) > m.$$

The number of variables is

$$\phi(n, m) + m^2 > m^2 + m.$$

Of these $m(m+1)/2$ are the functions $h_{i,k}$, hence of the variables at least

$$\phi(nm) + m^2 - \frac{m(m+1)}{2} > \frac{m(m+1)}{2}$$

are among the functions $\Lambda_{ik,\,lj}$.

Consider now that part of the new equations $\zeta_{r/i}$ which arises from the functions h_{ik} and $\Lambda_{ik,\,ik}$. We have seen that these latter functions may always be included among the new variables. Let us now put

$$B_{ik}^{1} = B_{ik}^{2} = \cdots = B_{ik}^{n} = B_{ik} \qquad (i, k = 1, 2, \cdots m).$$

Then

$$\Lambda_{ik,\,lj} = B_{ik} B_{lj} \left\{ \sum_{r=1}^{n}\sum_{s=1}^{n} a_{rs} - \sum_{r=1}^{m}\sum_{s=1}^{m} H_{rs} \right\}.$$

The expression in the brace is common to all the Λ's. It does not identically vanish, for in the present special case the first sum involves a_{nn} while the second does not. The a's may therefore be so taken that the expression does not vanish. Denote its value by M. Then

$$\Lambda_{ik,\,lj} = M B_{ik} B_{lj},$$

but

$$Y(\Lambda_{il,\,il}) = -4\sum_{r=1}^{m} \zeta_{r\,i}\, \Lambda_{ri,\,il},$$

and

$$Y(\Lambda_{ik,\,ik}) = -2\sum_{r=1}^{m} (\zeta_{r\,i}\, \Lambda_{rk,\,ik} + \zeta_{r\,k}\, \Lambda_{ir,\,ik}).$$

Hence, in the present case,

$$Y(\Lambda_{ii,\,ii}) = -4M \sum_{r=1}^{m} \zeta_{r\,i} B_{ri} B_{ii},$$

$$Y(\Lambda_{ik,\,ik}) = -2M \sum_{r=1}^{m} (\zeta_{r\,i} B_{rk} B_{ik} + \zeta_{r\,k} B_{ir} B_{ik}).$$

Similarly

$$Y(h_{ii}) = -2 \sum_{r=1}^{m} \zeta_{r\,i} h_{ri},$$

and

$$Y(h_{ik}) = -\sum_{r=1}^{m} (\zeta_{r\,i} h_{rk} + \zeta_{r\,k} h_{ri}),$$

which in this case become

$$Y(h_{ii}) = -2 \sum_{r=1}^{m} \zeta_{r\,i} A_{ri},$$

$$Y(h_{ik}) = -\sum_{r=1}^{m} (\zeta_{r\,i} A_{rk} + \zeta_{r\,k} A_{ir}).$$

If we now form that part of the matrix of the coefficients of the equations $\zeta_{r/i}$ which arises from the functions h_{ik} and $\Lambda_{ik,\,ik}$ we see that, first, from any column, such as that of the coefficients of $\partial f / \partial \Lambda_{ik,\,ik}$ we may divide out the common factor $2MB_{ik}$, and second, after this division the form of this part of the matrix is that of the matrix of the equations for the simultaneous invariants of order zero for the two forms

$$\phi_1 = \sum_{r=1}^{m} \sum_{s=1}^{m} A_{rs}\, dx_r\, dx_s,$$

$$\phi_2 = \sum_{r=1}^{m} \sum_{s=1}^{m} B_{rs}\, dx_r\, dx_s.$$

But we have seen * that it is then possible so to take the B_{rs} that there shall be a non-vanishing determinant of order m^2 in this matrix.

It therefore follows that the matrix of the equations of the present problem contains at least one determinant of order m^2 which does not in this special case vanish. Hence it does not *identically* vanish. *Hence the equations* $\zeta_{r/i}$ *are all independent.*

As the equations are all independent the *number of parameters is the excess of variables over equations* and the theorem is proved.

It may be noted that the choice of the quantities $B_{r/ik}$ by means of which the independence of the equations $\zeta_{r/i}$ is proved, is such as to make the Jacobian J vanish. This does not however invalidate the reasoning. For let K be the non-vanishing determinant of the matrix; J and K are both polynomials in the vari-

* Cf. note (§ 4), also *Invariants*, I, p. 91.

ables $y_{r/ik}$, and neither vanishes identically, for we have seen that it is possible to make either of them take a value different from zero. Hence it is possible to find values of the $y_{r,ik}$ such that they do not vanish simultaneously.

§ 7. *Parameters of order greater than* 2.

In the case of parameters of the first type, and also in that of parameters of the second type, provided the first method of determining the equations is used, the equations for the parameters of order μ differ from those for the invariants of order $\mu - 1$ only by the addition of terms. Consequently if the equations for the invariants of order $\mu - 1$ are independent, so are those for the parameters of order μ. But if $n > 2$ the equations for the invariants of order $\mu \geqq 2$ are all independent. Hence the equations for the parameters of order $\mu \geqq 3$ are all independent. Hence:

The number of parameters, of either type, of order not exceeding μ, *is, if* $\mu \geqq 3$ *equal to the excess of the number of variables over the number of equations determining those parameters. The cases* $\mu = 1, 2$ *have been treated separately.*

Suppose now $\mu \geqq 3$. The equations are all independent. The number of equations is, for parameters of either type, equal to the number of derivatives of the ξ's of orders not exceeding μ. The variables in the equations are of two kinds; first the a_{ik}'s and their derivatives of orders not exceeding $\mu - 1$; and, second, for parameters of the first type, the derivatives of the m functions U with respect to the n variables x; or, for parameters of the second type, the derivatives of the $n - m$ dependent, with respect to the m independent x's; the derivatives in both cases being of all orders not exceeding μ.

Hence the number of equations is, in both cases,

$$n\left\{\frac{(n+\mu)!}{n!\,\mu!} - 1\right\}.$$

The number of variables is, for parameters of the first type,

$$\frac{n(n+1)}{2}\frac{(n+\mu-1)!}{n!(\mu-1)!} + m\left\{\frac{(n+\mu)!}{n!\,\mu!} - 1\right\},$$

and, for parameters of the second type,

$$\frac{n(n+1)}{2}\frac{(n+\mu-1)!}{n!(\mu-1)!} + (n-m)\left\{\frac{(m+\mu)!}{m!\,\mu!} - 1\right\}.$$

The numbers of solutions are, then,

$$\frac{n(n+1)}{2}\frac{(n+\mu-1)!}{n!(\mu-1)!} - n\left\{\frac{(n+\mu)!}{n!\,\mu!} - 1\right\} + m\left\{\frac{(n+\mu)!}{n!\,\mu!} - 1\right\},$$

and

$$\frac{n(n+1)}{2}\frac{(n+\mu-1)!}{n!(\mu-1)!}-n\left\{\frac{(n+\mu)!}{n!\mu!}-1\right\}+(n-m)\left\{\frac{(m+\mu)!}{m!\mu!}-1\right\},$$

in the two cases respectively.

But of these solutions,

$$\frac{n(n+1)}{2}\frac{(n+\mu-1)!}{n!(\mu-1)!}-n\left\{\frac{(n+\mu)!}{n!\mu!}-1\right\}$$

are *invariants* of order $\leqq \mu-1$.

Hence the number of *parameters* of order $\leqq \mu$ and of the first type, is

$$m\left\{\frac{(n+\mu)!}{n!\mu!}-1\right\},$$

and of the second type

$$(n-m)\left\{\frac{(m+\mu)!}{m!\mu!}-1\right\}.$$

If $\mu \geqq 4$ then the corresponding relations hold for $\mu-1$.

Hence the number of parameters of order μ of the first type is

$$\frac{m(n+\mu-1)!}{(n-1)!\,\mu!},$$

and of the second type

$$\frac{(n-m)(m+\mu-1)!}{(m-1)!\,\mu!}.$$

Here $\mu \geqq 4$, $n \geqq 3$.

The number of parameters of orders not exceeding 3 and of the first type is

$$m\left\{\frac{(n+1)(n+2)(n+3)}{6}-1\right\}=\frac{mn(n+1)(n+2)}{6}+\frac{mn(n+3)}{2},$$

and of the second type

$$(n-m)\left\{\frac{(m+1)(m+2)(m+3)}{6}-1\right\}=\frac{(n-m)m(m+1)(m+2)}{6}+(n-m)\frac{m(m+3)}{2}.$$

The number of parameters of orders not exceeding 2 and of the first type is

$$\frac{mn(n+3)}{2}-\frac{n(n-1)}{2},$$

and of the second type

$$\frac{m(m+3)(m^2-m+2)}{8}, \qquad \text{if } \frac{m(m+3)}{2} \leqq n,$$

$$(n-m)\frac{m(m+3)}{2}-\frac{n(n-1)}{2}, \qquad \text{if } \frac{m(m+3)}{2}\geqq n,$$

Hence *the number of parameters of order* 3 *and of the first type is*

$$\frac{mn(n+1)(n+2)}{6}+\frac{n(n-1)}{2},$$

and of the second type

$$(n-m)\frac{m(m+1)(m+2)}{6}+\frac{m(m+3)}{2}\left\{n-\frac{(m+1)(m+2)}{4}\right\},$$

$$\text{if } \frac{m(m+3)}{2}\leqq n,$$

and

$$(n-m)\frac{m(m+1)(m+2)}{6}+\frac{n(n-1)}{2}, \qquad \text{if } \frac{m(m+3)}{2}\geqq n.$$

SHEFFIELD SCIENTIFIC SCHOOL OF YALE UNIVERSITY,
August, 1903.

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The case $n=2$ has not been here treated, for ŻORAWSKI, and more recently, FORSYTH,* have already discussed it in full. The methods here used apply however to this case also, and effect a considerable simplification of the work. For example for the determination of the parameters of the second type and order we have the single equation

$$2h_{11}\frac{\partial f}{\partial h_{11}}+4\Lambda_{11,11}\frac{\partial f}{\partial \Lambda_{11,11}}=0.$$

The parameter sought is therefore

$$P\equiv\frac{\Lambda_{11,11}}{h_{11}^2}$$

which is readily seen to be the square of the geodesic curvature of a curve on the surface whose squared linear element is given by the differential form under consideration.

The functions $y_{1/11}, y_{2/11}$ have a simple geometric interpretation, for the equations

$$\underset{11}{y_1}=0, \qquad \underset{11}{y_2}=0$$

are,† if z_1, the independent variable, is the length of arc of a geodesic line, the equations which determine the geodesics on the surface.

*FORSYTH, Philosophical Transactions, vol. 201 (1903), A, pp. 329–402.

†Cf. BIANCHI, *Differentialgeometrie*, p. 151.

ON THE COEFFICIENTS IN THE PRODUCT OF AN ALTERNANT AND A SYMMETRIC FUNCTION*

BY

E. D. ROE, JR.

§1. *Introduction. Notations.*

If with respect to m indeterminates $\alpha_1, \cdots, \alpha_m$ A denote the simplest alternant $|0\ 1 \cdots m-1|$, viz. $|\alpha_1^0 \alpha_2^1 \cdots \alpha_m^{m-1}|$, ϕ a monomial symmetric function, and A_k an alternant in their product

$$A\phi = C_1 A_1 + C_2 A_2 + \cdots + C_s A_s,^{**} \tag{1}$$

it was formerly necessary to find all of the C's in order to find one of them. The problem of obtaining a single C alone was first considered by MUIR † in a brief paper in 1899. His method was first to express ϕ by means of the symmetric function table in terms of products p of weight w of elementary symmetric functions, as

$$\phi = c_1 p_1 + c_2 p_2 + \cdots + c_r p_r. \tag{2}$$

This changes the problem to that of finding the d's in

$$Ap_k = d_{k,1} A_1 + d_{k,2} A_2 + \cdots + d_{k,s} A_s. \tag{3}$$

If found, the C's are expressed by the relations

$$C_k = c_1 d_{1,k} + c_2 d_{2,k} + \cdots + c_r d_{r,k}. \tag{4}$$

MUIR obtained the expression for one particular C,‡ in terms of the c's. Professor TAYLOR, proceeding from the standpoint of MUIR, has extended the solution, so as to obtain any C in terms of the c's and d's.§ Another method

* Presented to the Society December 29, 1902. Received for publication November 16, 1903.

** Cf. MUIR, *Theory of Determinants*, p. 176, §129.

† *The Multiplication of an Alternant by a Symmetric Function of the Variables*, Proceedings of the Royal Society of Edinburgh, 1898–99, pp. 539–542.

‡ It is the relation (32) obtained in this paper.

§ *On the Product of an Alternant by a Symmetric Function*, American Mathematical Monthly, vol. 10 (1903), pp. 119–130.

of attacking this problem is that given by the writer.* It determines the C's directly from the original function without recourse to the elementary products and the symmetric function table. In fact *it becomes simultaneously a method for expressing symmetric functions in terms of the elementary products.* These two results were reached formally in a theorem

$$(5) \qquad |012\cdots n-1|\sum \alpha_1^{p_1}\alpha_2^{p_2}\cdots\alpha_m^{p_m} = \sum \begin{Bmatrix} p_1 p_2\cdots p_m \\ i_1 i_2\cdots i_m \end{Bmatrix} |i_1 i_2\cdots i_m|,*$$

and in a related theorem,

$$(6) \qquad a_0^n \sum \alpha_1^{p_1}\alpha_2^{p_2}\cdots\alpha_m^{p_m} = (-1)^r \sum \begin{Bmatrix} p_1 p_2\cdots p_m \\ i_1 i_2\cdots i_m \end{Bmatrix} \{i_{m+1} i_{m+2}\cdots i_{m+n}\}.*$$

(The notations are to be explained immediately.) The latter expresses a monomial symmetric function as a sum of determinants in the a's, the elementary symmetric functions. These results were obtained by using the resultant of two binary forms and the theory of corresponding matrices.†

In this paper, which may be regarded as a continuation of my paper just cited, occur among other things: Direct proofs, without the use of the resultant, and for more general results when the alternant $|\lambda_1\lambda_2\cdots\lambda_m|$ is used as multiplier, §§ 2 and 3; explicit methods for calculation and a generalized expression for the coefficients when the multiplier is $|012\cdots m-1|$, §§ 3–6; the C's in terms of the c's and the d's, which latter are also proved to be the coefficients in the quotient of two alternants, § 4. It is proved that a table of C's for weight w may be so arranged that it fills out a right triangle whose hypothenuse consists of ones, and the sum of whose every column except the last is zero, § 7. The d's are expressed in terms of the C's, § 8. The absolute invariance of the C's is proved when $|012\cdots m+r-1|$ is the multiplier, § 9. And it is shown, § 10, how a two-fold application of the table gives any monomial symmetric function in terms of elementary products, so that this table is, in a certain sense, a square root of the usual symmetric function table.

A table of the C's for weight w is called a product table for weight w. The theory involves certain general complexes of integers: $\lambda_1\lambda_2\cdots\lambda_m$, $\kappa_1\kappa_2\cdots\kappa_m$, $i_1 i_2\cdots i_m$, $p_1 p_2\cdots p_m$, etc.; these are denoted by λ, κ, i, p, etc., while the complex $\kappa_{i_1}\kappa_{i_2}\cdots\kappa_{i_m}$ is denoted by κ_i. The conjugate complexes $i_1 i_2\cdots i_m$, and $i_{m+1} i_{m+2}\cdots i_{m+n}$ formed from each other, as previously explained, l. c.,* by taking those elements of the series $012\cdots m+n-1$ which remain after the complementaries of the given elements with respect to $m+n-1$ have

* *Note on Symmetric Functions*, American Journal of Mathematics, vol. 25 (1903), pp. 97–106.

† GORDAN, *Invariantentheorie*, vol. 1, p. 95.

been rejected, are denoted by i and $\bar{i}$. We have $\bar{\bar{i}} = i$. If $n = m$, i and $\bar{i}$ are principal conjugates according to § 7, 1.

The functions and the coefficients to be studied have the following definitions and notations:

$$(7) \qquad \begin{vmatrix} \alpha_1^{\kappa_1} & \alpha_1^{\kappa_2} & \cdots & \alpha_1^{\kappa_m} \\ \alpha_2^{\kappa_1} & \alpha_2^{\kappa_2} & \cdots & \alpha_2^{\kappa_m} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_m^{\kappa_1} & \alpha_m^{\kappa_2} & \cdots & \alpha_m^{\kappa_m} \end{vmatrix} = |\kappa_1 \kappa_2 \cdots \kappa_m| = |\kappa| ;$$

$$|\kappa_{i_1} \kappa_{i_2} \cdots \kappa_{i_m}| = |\kappa_i| .$$

$$(8) \qquad \begin{vmatrix} a_{\kappa_1-\lambda_1} & a_{\kappa_2-\lambda_1} & \cdots & a_{\kappa_m-\lambda_1} \\ a_{\kappa_1-\lambda_2} & a_{\kappa_2-\lambda_2} & \cdots & a_{\kappa_m-\lambda_2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{\kappa_1-\lambda_m} & a_{\kappa_2-\lambda_m} & \cdots & a_{\kappa_m-\lambda_m} \end{vmatrix} = \{\kappa_1 - \lambda_1 \ \kappa_2 - \lambda_1 \cdots \kappa_m - \lambda_1\} = \{\kappa - \lambda\}$$

$$= [\kappa_1 - \lambda_1 \ \kappa_2 - \lambda_2 \cdots \kappa_m - \lambda_m] .$$

$$(9) \qquad \sum \alpha_1^{p_1} \alpha_2^{p_2} \cdots \alpha_m^{p_m} = (p_1 p_2 \cdots p_m) = (p),$$

$$(\sum \alpha_1 \alpha_2 \cdots \alpha_{j_1})^{i_1} = (1^{j_1})^{i_1},$$

$$(1^{j_1})^{i_1} (1^{j_2})^{i_2} \cdots (1^{j_m})^{i_m} = (1^j)^i .$$

$$(10) \qquad \begin{vmatrix} \kappa_1 \kappa_2 \cdots \kappa_m \\ \lambda_1 \lambda_2 \cdots \lambda_m \ p_1 p_2 \cdots p_m \end{vmatrix} = \begin{vmatrix} \kappa \\ \lambda \ p \end{vmatrix} = \text{the coefficient of } |\kappa| \text{ in } |\lambda|(p).$$

$$(11) \qquad \begin{vmatrix} \kappa_1 \kappa_2 \cdots \kappa_m \\ \lambda (1^{j_1})^{i_1} (1^{j_2})^{i_2} \cdots (1^{j_m})^{i_m} \end{vmatrix} = \begin{vmatrix} \kappa \\ \lambda (1^j)^i \end{vmatrix} = \text{the coefficient of } |\kappa| \text{ in } |\lambda| (1^j)^i .$$

$$(12) \qquad \begin{vmatrix} p_1 p_2 \cdots p_m \\ \kappa_1 \kappa_2 \cdots \kappa_m / \lambda_1 \lambda_2 \cdots \lambda_m \end{vmatrix} = \begin{vmatrix} p \\ \kappa / \lambda \end{vmatrix} = \text{the coefficient of } (p) \text{ in } |\kappa| / |\lambda| .$$

$$(13) \qquad \left\{ \begin{matrix} p_1 p_2 \cdots p_m \\ \kappa_1 - \lambda_1 \ \kappa_2 - \lambda_1 \cdots \kappa_m - \lambda_1 \end{matrix} \right\} = \left[\begin{matrix} p_1 p_2 \cdots p_m \\ \kappa_1 - \lambda_1 \ \kappa_2 - \lambda_2 \cdots \kappa_m - \lambda_m \end{matrix} \right]$$

$$= \left\{ \begin{matrix} p \\ \kappa - \lambda \end{matrix} \right\} = \text{the coefficient of } a_{p_1} a_{p_2} \cdots a_{p_m} = a_p \text{ in } \{\kappa - \lambda\} .$$

$$\begin{pmatrix} p_1 p_2 \cdots p_m \\ 0^m \ \kappa_1 \kappa_2 \cdots \kappa_m \end{pmatrix} = \begin{pmatrix} p \\ \kappa \end{pmatrix} = \text{the coefficient of } a_{p_1} a_{p_2} \cdots a_{p_m} = a_p \text{ in } a_0^m(\kappa) .$$

The notations in (10), (11) and (12) will usually be written in this paper with the complex λ omitted, the known and definite λ so omitted being in each case specified in the context. Thus (10) and (12) are written $\left|\begin{smallmatrix} \kappa_1 \kappa_2 \cdots \kappa_m \\ p_1 p_2 \cdots p_m \end{smallmatrix}\right| = \left|\begin{smallmatrix} \kappa \\ p \end{smallmatrix}\right|$, $\left\|\begin{smallmatrix} p_1 p_2 \cdots p_m \\ \kappa_1 \kappa_2 \cdots \kappa_m \end{smallmatrix}\right\| = \left\|\begin{smallmatrix} p \\ \kappa \end{smallmatrix}\right\|$. In the former the weight of the upper, in the latter that of the lower indices, is greater.

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$$(10)\qquad \left| \begin{matrix} \kappa_1 \kappa_2 \cdots \kappa_m \\ \lambda_1 \lambda_2 \cdots \lambda_m \ p_1 p_2 \cdots p_m \end{matrix} \right| = \left| \begin{matrix} \kappa \\ \lambda \, p \end{matrix} \right| = \text{the coefficient of } |\kappa| \text{ in } |\lambda|(p).$$

$$(11)\qquad \left| \begin{matrix} \kappa_1 \kappa_2 \cdots \kappa_m \\ \lambda (1^{j_1})^{i_1} (1^{j_2})^{i_2} \cdots (1^{j_m})^{i_m} \end{matrix} \right| = \left| \begin{matrix} \kappa \\ \lambda (1^j)^i \end{matrix} \right| = \text{the coefficient of } |\kappa| \text{ in } |\lambda| (1^j)^i .$$

$$(12)\qquad \left\| \begin{matrix} p_1 p_2 \cdots p_m \\ \kappa_1 \kappa_2 \cdots \kappa_m / \lambda_1 \lambda_2 \cdots \lambda_m \end{matrix} \right\| = \left\| \begin{matrix} p \\ \kappa / \lambda \end{matrix} \right\| = \text{the coefficient of } (p) \text{ in } |\kappa| / |\lambda| .$$

$$(13)\qquad \left\{ \begin{matrix} p_1 p_2 \cdots p_m \\ \kappa_1 - \lambda_1 \ \kappa_2 - \lambda_1 \cdots \kappa_m - \lambda_1 \end{matrix} \right\} = \left[\begin{matrix} p_1 p_2 \cdots p_m \\ \kappa_1 - \lambda_1 \ \kappa_2 - \lambda_2 \cdots \kappa_m - \lambda_m \end{matrix} \right]$$

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$$\left(\begin{matrix} p_1 p_2 \cdots p_m \\ 0^m \ \kappa_1 \kappa_2 \cdots \kappa_m \end{matrix} \right) = \left(\begin{matrix} p \\ \kappa \end{matrix} \right) = \text{the coefficient of } a_{p_1} a_{p_2} \cdots a_{p_m} = a_p \text{ in } a_0^m(\kappa) .$$

The notations in (10), (11) and (12) will usually be written in this paper with the complex λ omitted, the known and definite λ so omitted being in each case specified in the context. Thus (10) and (12) are written $\left| \begin{matrix} \kappa_1 \kappa_2 \cdots \kappa_m \\ p_1 p_2 \cdots p_m \end{matrix} \right| = \left| \begin{matrix} \kappa \\ p \end{matrix} \right|$, $\left\| \begin{matrix} p_1 p_2 \cdots p_m \\ \kappa_1 \kappa_2 \cdots \kappa_m \end{matrix} \right\| = \left\| \begin{matrix} p \\ \kappa \end{matrix} \right\|$. In the former the weight of the upper, in the latter that of the lower indices, is greater.

§ 2. *The Theorem* $\left|\begin{smallmatrix}\kappa\\ \lambda p\end{smallmatrix}\right| = \left\{\begin{smallmatrix}p\\ \kappa-\lambda\end{smallmatrix}\right\}$.

We shall prove that $\left|\begin{smallmatrix}\kappa\\ \lambda p\end{smallmatrix}\right| = \left\{\begin{smallmatrix}p\\ \kappa-\lambda\end{smallmatrix}\right\}$. This is seen by considering the mode of formation of $\left|\begin{smallmatrix}\kappa\\ \lambda p\end{smallmatrix}\right|$. In order to obtain it we add a permutation p_i of the p's to λ and get,

$$p_{i_1}+\lambda_1,\ p_{i_2}+\lambda_2,\ \cdots,\ p_{i_m}+\lambda_m. \tag{14}$$

If these numbers are all different and a permutation of κ, we have one of the alternants sought, namely

$$|\kappa_i| = (-1)^i|\kappa|, \tag{15}$$

where

$$p_{i_r}+\lambda_r=\kappa_{i_r}, \qquad \text{or} \qquad p_{i_r}=\kappa_{i_r}-\lambda_r \qquad (r=1, 2, \cdots m), \tag{16}$$

and where the exponent i is the number of inversions of order in $\kappa_i=\kappa_{i_1}\kappa_{i_2}\cdots\kappa_{i_m}$ from the order $\kappa=\kappa_1\kappa_2\cdots\kappa_m$ of ascending magnitude, i. e., i is the number of inversions of order among $i_1, i_2, \cdots i_m$. Then it is evident that

$$\left|\begin{matrix}\kappa\\ \lambda p\end{matrix}\right| = \sum(-1)^i. \tag{17}$$

The number of terms in $\sum$ is the number of complete sets of solutions of the equations (16). To define this number we write

$$\begin{matrix}\kappa_1-\lambda_1 & \kappa_2-\lambda_1 & \cdots & \kappa_m-\lambda_1\\ \kappa_1-\lambda_2 & \kappa_2-\lambda_2 & \cdots & \kappa_m-\lambda_2\\ \cdot & \cdot & \cdot & \cdot\\ \kappa_1-\lambda_m & \kappa_2-\lambda_m & \cdots & \kappa_m-\lambda_m\end{matrix} \qquad (m \text{ lines}) \tag{18}$$

and observe that according to (16) each κ in the complex κ_i comes from the addition of a $\kappa-\lambda_i$ in the ith line of the array (18), to the ith number in the complex λ, provided that $\kappa-\lambda_i$ is one of the p's. Then to form a permutation of the κ's, only one number can be added from the ith line to λ_i, and also only one number can be taken from the jth column. Hence a permutation of $\kappa_1\kappa_2\cdots\kappa_m$ can be formed in as many ways as we can select one and only one number from each line and column of the array (18), *with the restriction* that the numbers selected form a permutation of the p's. Let the element selected from the rth line be $\kappa_{i_r}-\lambda_r$. Then the selected elements correspond to the substitution $\left(\begin{smallmatrix}1&2&3&\cdots&m\\ i_1&i_2&i_3&\cdots&i_m\end{smallmatrix}\right)$ with modulus $(-1)^i$. If we select a term a_p from the determinant $\{\kappa-\lambda\}$, one such term would, from the foregoing, evidently be obtained by selecting the elements from the determinant according to the substitution $\left(\begin{smallmatrix}1&2&3&\cdots&m\\ i_1&i_2&i_3&\cdots&i_m\end{smallmatrix}\right)$ and the sign of this term would also be $(-1)^i$. Hence to every alternant $|\kappa_i|=(-1)^i|\kappa|$, corresponds a term $(-1)^i a_p$ of the determinant $\{\kappa-\lambda\}$, and conversely, and hence

$$\left\{ \begin{matrix} p \\ \kappa - \lambda \end{matrix} \right\} = \sum (-1)^t \tag{19}$$

and therefore by (17)

$$\begin{vmatrix} \kappa \\ \lambda p \end{vmatrix} = \left\{ \begin{matrix} p \\ \kappa - \lambda \end{matrix} \right\}. \tag{20}$$

If $\lambda = 012 \cdots m-1$, this relation becomes

$$\begin{vmatrix} \kappa \\ p \end{vmatrix} = \left\{ \begin{matrix} p \\ \kappa \end{matrix} \right\}.$$

§ 3. *On the calculation of* $\left|\begin{smallmatrix} \kappa \\ \lambda p \end{smallmatrix}\right|$.

1. In the paper previously cited (p. 2) a reduction formula was given for coefficients when the multiplier is $|012 \cdots m-1|$. It is easy to see what this formula becomes for the general case. Put $\kappa_r - \lambda_1 = \sigma_r$. If r numbers be common to the two complexes p, σ, so that

$$\sigma_{\mu_1} = p_{\nu_1},\ \sigma_{\mu_2} = p_{\nu_2},\ \cdots,\ \sigma_{\mu_r} = p_{\nu_r}, \tag{21}$$

we have by expanding $\{\sigma\}$ in terms of the elements of its first line

$$\left\{ \begin{matrix} p \\ \sigma \end{matrix} \right\} = \sum_{\kappa=1}^{\kappa=r} (-1)^{\mu_\kappa - 1} \left\{ \begin{matrix} p_{\nu_1} & p_{\nu_2} & \cdots p_{\nu_{\kappa-1}} & p_{\nu_{\kappa+1}} & \cdots p_{\nu_m} \\ \sigma_1 - \tau & \sigma_2 - \tau \cdots \sigma_{\mu_{\kappa-1}} - \tau & \sigma_{\mu_{\kappa-1}} - \tau \cdots \sigma_m - \tau \end{matrix} \right\}, \tag{22}$$

where $\tau = \lambda_2 - \lambda_1$. In general in the rth stage of reduction we subtract $\lambda_{r+1} - \lambda_r$ from all the elements remaining in the lower line. If no numbers of the two series are common, or if a lower index becomes negative, the coefficient is zero. For example, to find $\left|\begin{smallmatrix} 2459 \\ 0^2 13 \end{smallmatrix}\right|$ with $\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 5$, $\lambda_4 = 6$, we have,

$$\begin{vmatrix} 2459 \\ 0^2 13 \end{vmatrix} = \left\{ \begin{matrix} 0^2 13 \\ 0237 \end{matrix} \right\} = \left\{ \begin{matrix} 013 \\ 126 \end{matrix} \right\} = \left\{ \begin{matrix} 03 \\ 04 \end{matrix} \right\} = \left\{ \begin{matrix} 3 \\ 3 \end{matrix} \right\} = 1. \tag{23}$$

2. The calculation is shortened by changing the notation slightly. Instead of the first line, the principal diagonal indices κ_1, $\kappa_2 - 1$, $\cdots$, $\kappa_m - m + 1$, (for $\lambda = 012 \cdots m-1$) may be used. They are the different partitions of weight m. Thus for weight 5 we have the partitions:

00005
00014
00023
00122
01112
11111

Taking each one with all the others in order from below upwards, one obtains all the coefficients of the table for weight 5. Thus the first one would be

$$(24) \qquad \left[\begin{matrix}00005\\11111\end{matrix}\right] = \left\{\begin{matrix}00005\\12345\end{matrix}\right\} = \left|\begin{matrix}12345\\00005\end{matrix}\right|,$$

where $[11111] = \{12345\}$.

The value of $\left[\begin{matrix}p_1p_2\cdots p_m\\i_1i_2\cdots i_m\end{matrix}\right]$ is the number of ways in which the lower complex, i, can be made equal to the upper, p, by the following operation: passing a lower number from right to left over an adjacent number introduces the factor (-1) towards the result, reduces the number passed over by one, and increases that passing over by one. The coefficient is the algebraic sum of the number of *distinct ways* in which the complex i can thus be made equal to the complex p. E. g.

$\left[\begin{matrix}00005\\11111\end{matrix}\right] = (-1)^4$. The right hand unit, passed over the preceding four, produces four zeros and a five; and this result can be produced in no other way by the process described.

$\left[\begin{matrix}00113\\11111\end{matrix}\right] = 3$. Here, as two zeros and a three must be obtained, unity must be passed over two units. This can be done in three ways, by passing the 3d, 4th or 5th unit over two preceding ones.

$\left[\begin{matrix}00122\\11111\end{matrix}\right] = 3$. (2d and 4th) or (3d and 5th) or (2d and 5th) over one preceding unit.

$\left[\begin{matrix}00014\\11111\end{matrix}\right] = -2$. 4th or 5th over three preceding units.

The theory of this is simply that the operations performed represent the *different ways* in which the term in question can be brought to the principal diagonal of the determinant involved, and it becomes clear by considering the effect on the indices when one column is passed over another at its left.

If $|\lambda|$ is used as multiplier, the factor (-1) is introduced as before, but if the element in the rth place pass over that in the $(r-1)$th, that passing over is increased, while that passed over is decreased by $\lambda_r - \lambda_{r-1}$. For example with $\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 5$, $\lambda_4 = 6$, $\left|\begin{matrix}2567\\0013\end{matrix}\right| = \left\{\begin{matrix}0013\\0345\end{matrix}\right\} = \left[\begin{matrix}1013\\0211\end{matrix}\right] = -1$, for in moving 1 in the third place over 2, the former must be increased and the latter diminished by 2, and this is the only way of making the two complexes alike. It is to be especially noted that the difference $\lambda_r - \lambda_{r-1}$ *belongs to the rth place* whatever the elements moved.

§ 4 *Relations between the Coefficients* $\left|\begin{matrix}*\\p\end{matrix}\right|$ *and the Coefficients in the Symmetric Function Table.*

In the paper already cited (p. 2) the relation

$$\left(\begin{matrix}q_1q_2\cdots q_n\\0^n p_1p_2\cdots p_m\end{matrix}\right) = (-1)^v \sum_i \left|\begin{matrix}i_1i_2\cdots i_m\\p_1p_2\cdots p_m\end{matrix}\right| \left|\begin{matrix}i_{m+1}\cdots i_{m+n}\\q_1q_2\cdots q_n\end{matrix}\right|$$

or,

$$\left(\begin{matrix}q\\p\end{matrix}\right) = (-1)^v \sum_i \left|\begin{matrix}i & \bar{i}\\p & q\end{matrix}\right|,$$

where p and q are of weight w, i of weight $w + \frac{1}{2}m(m-1)$, and $\bar{\imath}$ of weight $w + \frac{1}{2}n(n-1)$, was given.

We shall now express $\left|\begin{smallmatrix} i \\ \rho \end{smallmatrix}\right|$ in terms of the $\binom{p}{\kappa}$ and the $\left\|\begin{smallmatrix} \kappa \\ \bar{\imath} \end{smallmatrix}\right\|$, where the complexes p and κ are of weight w, i is of weight $w + \frac{1}{2}m(m-1)$, and $\bar{\imath}$ is of weight $w + \frac{1}{2}n(n-1)$.

By using the theorem of corresponding matrices and by the use of the matrices employed in the former paper and the additional matrix

$$\begin{vmatrix} \beta_1^{2m-1} & \beta_1^{2m-2} & \cdots & \beta_1 & 1 \\ \beta_2^{2m-1} & \beta_2^{2m-2} & \cdots & \beta_2 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \beta_n^{2m-1} & \beta_n^{2m-2} & \cdots & \beta_n & 1 \end{vmatrix} \qquad (n \text{ lines}),$$

where the β's are the roots of

$$\phi(x) = b_0 x^n + b_1 x^{n-1} + \cdots + b_n = 0,$$

we have, for $\lambda = 012 \cdots m-1$,

$$\{i\} = (-1)^{\mu} \sigma \left| i_{m+n} i_{m+n-1} \cdots i_{m+1} \right|, \tag{25}$$

where

$$\mu = \frac{m(m+1)}{2} + i_1 + 1 + i_2 + 1 + \cdots + i_m + 1 = m(m+1) + w,$$
$$\equiv w \pmod 2.$$

Also

$$\sigma = \frac{b_0^m}{|n-1, n-2, \cdots, 1, 0|},$$

therefore

$$\{i\} = (-1)^w b_0^m \frac{\left| i_{m+1} i_{m+2} \cdots i_{m+n} \right|}{|0, 1, \cdots, n-1|}, \tag{26}$$

and hence

$$\left\{ \begin{matrix} p \\ i \end{matrix} \right\} = \text{the coefficient of } b_\rho \text{ in } (-1)^w b_0^m \frac{\left| i_{m+1} \cdots i_{m+n} \right|}{|0, 1, \cdots, n-1|}. \tag{27}$$

Since

$$i_{m+1} + \cdots + i_{m+n} = \frac{n(n-1)}{2} + w,$$

$\left| i_{m+1} \cdots i_{m+n} \right| / |0, 1, \cdots, n-1|$ is a linear function of monomial symmetric functions of weight w, and from the quotient of this division one infers directly the expression for $\left|\begin{smallmatrix} i \\ \rho \end{smallmatrix}\right|$ as a linear function of the coefficients in the different symmetric functions of weight w (cf. § 8). Hence

$$(28)\qquad \frac{|i_{m+1}\cdots i_{m+n}|}{|0,1,\cdots,n-1|} = \sum_{\kappa} \begin{vmatrix} \kappa_1\kappa_2\cdots\kappa_n \\ i_{m+1}\cdots i_{m+n}\end{vmatrix} (\kappa_1\kappa_2\cdots\kappa_n),$$

and since $\binom{p}{\kappa} = \binom{\kappa}{p}$,* we have

$$(29)\qquad \begin{vmatrix} i \\ p\end{vmatrix} = (-1)^w \sum_{\kappa}\binom{p}{\kappa} \begin{vmatrix} \kappa \\ \bar{\imath}\end{vmatrix} = (-1)^w \sum_{\kappa}\binom{\kappa}{p} \left\| \begin{matrix} \kappa \\ \bar{\imath}\end{matrix} \right\|,$$

where the complex κ begins with i_{m+1}, $i_{m+2}-1, \cdots i_{m+n}-n+1$, and then runs through each succeeding partition (containing n elements) † in the sense of § 7, 2. A table containing coefficients $\left\| \begin{matrix} \kappa \\ \bar{\imath}\end{matrix} \right\|$ for weight w, is called a quotient table for weight w. ‡ What has just preceded shows that Dr. TAYLOR's product table is the author's quotient table, with conjugate columns interchanged, § for it gives the multiplier $\left\| \begin{matrix} \kappa \\ \bar{\imath}\end{matrix} \right\|$ by which the symmetric function (κ) and therefore the individual coefficients in its expression in terms of elementary symmetric functions are to be multiplied in order to obtain $\begin{vmatrix} i \\ p\end{vmatrix}$. But this is the coefficient which Dr. TAYLOR seeks. The exact relation between the two, using the writer's notation, may therefore be expressed as follows:

$$(30)\qquad \begin{vmatrix} 1^{r_1}2^{r_2}\cdots n^{r_n} \\ i_{m+1}\cdots i_{m+n}\end{vmatrix} = \begin{vmatrix} i_1 i_2\cdots i_m \\ (1)^{r_1}(1^2)^{r_2}\cdots(1^n)^{r_n}\end{vmatrix},$$

where the coefficient on the left is a coefficient in the author's quotient table, that on the right, one in Dr. TAYLOR's product table. We have in fact

$$\begin{vmatrix} i \\ p\end{vmatrix} = (-1)^w \sum_{\kappa}\binom{\kappa}{p} \begin{vmatrix} i \\ (1^\kappa)\end{vmatrix},$$

by (4) and

$$\begin{vmatrix} i \\ p\end{vmatrix} = (-1)^w \sum_{\kappa}\binom{\kappa}{p} \begin{vmatrix} \kappa \\ \bar{\imath}\end{vmatrix},$$

by (29). Putting the two expressions for $\begin{vmatrix} i \\ p\end{vmatrix}$ equal to each other, and equating coefficients of $\binom{\kappa}{p}$ and writing κ in the form $1^{r_1}2^{r_2}\cdots n^{r_n}$ we have the relation (30), as stated. By using a theorem proved in § 7, 3, that the sum of the coefficients of every column in the table for weight w is zero, except for the last column, which contains only unity, it follows that in particular

$$\frac{|0,1,2,\cdots,m-2,\ 2m-1|}{|0,1,\ 2,\cdots,m-1|}$$

* This is the CAYLEY-BETTI law of symmetry. See CAYLEY, *Collected Mathematical Papers*, vol. 2, p. 421. The writer has given a proof of it in the American Mathematical Monthly, vol. 6 (1899), p. 104.

† The number n is the order of (p).

‡ The author reserves for a future paper a consideration of the properties of such coefficients similar to that of the present paper for the coefficients of the product table.

§ Since the complexes $i = i_1 i_2\cdots i_m$ and $\bar{\imath} = i_{m+1}\cdots i_{m+n}$ of (26) are conjugate to each other in the sense of § 7, 1.

equals the sum of all monomial symmetric functions of weight w. Hence for any coefficient in the first column we have,

$$\begin{vmatrix} 1\,2\,3\cdots m \\ p_1\,p_2\cdots p_m \end{vmatrix} = (-1)^w \sum_{\kappa} \binom{p}{\kappa} \quad (\kappa_1+\kappa_2+\cdots+\kappa_m = w), \tag{31}$$

or, by the CAYLEY-BETTI law,

$$\begin{vmatrix} 1\,2\,3\cdots m \\ p_1\,p_2\cdots p_m \end{vmatrix} = (-1)^w \sum_{\kappa} \binom{\kappa}{p}.^* \tag{32}$$

The reader may readily verify the foregoing discussion by reference to a table of symmetric functions in connection with the tables here given. Thus he will find:

$$\begin{vmatrix} 1234567 \\ 0^5 16 \end{vmatrix} = -(7-1-7-7+1+8+4-1+7-9+1-5+5-1) = -2.$$

In a similar manner we have by performing the divisions,

$$\frac{|01235\ 10|}{|012345|} = (51)+(42)+(3^2)+2(41^2)+2(321)+3(31^3) + 2(2^3)+3(2^21^2)+4(21^4)+5(1^6),$$

$$\frac{|012369|}{|012345|} = (42)+(3^2)+(41^2)+2(321)+3(31^3)+3(2^3) + 4(2^21^2)+6(21^4)+9(1^6),$$

and these give the multipliers for forming the coefficients, in the product, by the symmetric function table, in the second and third columns of our table for weight six. Thus: $\left|\begin{smallmatrix} 023457 \\ 0^4 15 \end{smallmatrix}\right|$ = coefficient of $b_0^4 b_1 b_5$ in

$$b_0^6 \frac{|01235\ 10|}{|012345|} = -1-6-3+2.1+2.7+3(-1)+2.2+3(-4)+4.1 = -1,$$

$$\begin{vmatrix} 013467 \\ 0^1 24 \end{vmatrix} = 2-3+2+2.4+3(-2)+3(-2)+4.1 = 1.$$

By § 3, 2, for confirmation and comparison of methods,

$$\begin{vmatrix} 023457 \\ 0^4 15 \end{vmatrix} = \begin{Bmatrix} 0^4 15 \\ 023457 \end{Bmatrix} = \begin{bmatrix} 000015 \\ 011112 \end{bmatrix} = -1,$$

the latter result, since 2 must pass over three elements to the left, and this is the only way of making both complexes equal. $\left|\begin{smallmatrix} 013467 \\ 0^4 24 \end{smallmatrix}\right| = \left\{\begin{smallmatrix} 0^4 24 \\ 013467 \end{smallmatrix}\right\} = \left[\begin{smallmatrix} 000024 \\ 001122 \end{smallmatrix}\right] = 1,$

* This is the relation obtained by MUIR. Its explicit expression is obtained in this paper in § 6, (44). See also § 6, (45).

since 2 in the lower complex passes over the two preceding units, to make the two complexes equal.

§ 5. *Relations between the t's and the a's.*

By t_w we denote the sum of all the monomial symmetric functions of weight w. Then one has

$$\sum_0^n a_r x^r = a_0 \prod_1^n (1 - \alpha_r x), \tag{33}$$

$$\frac{1}{\sum_0^n a_r x^r} = \frac{1}{a_0} \sum_0^\infty t_r x^r, \tag{34}$$

as identities in the a's, the α's and x. From (34), cleared of fractions, by equating the coefficients of x^n,

$$a_0 t_n + a_1 t_{n-1} + \cdots + a_n t_0 = 0,* \tag{35}$$

whence

$$a_0^w t_w = (-1)^w \begin{vmatrix} a_1 & a_2 & \cdots & a_w \\ a_0 & a_1 & \cdots & a_{w-1} \\ 0 & a_0 & \cdots & a_{w-2} \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & a_0 \; a_1 \end{vmatrix}. \tag{36}$$

By developing the left member of (34) again in terms of a's and x and equating coefficients of x^w, we have

$$a_0^w t_w = \sum (-1)^{\lambda_1+\lambda_2+\cdots+\lambda_w} \frac{(\lambda_1 + \lambda_2 + \cdots + \lambda_w)!}{\lambda_1!\,\lambda_2! \cdots \lambda_w!} a_0^{\lambda_0} a_1^{\lambda_1} \cdots a_w^{\lambda_w},\dagger \tag{37}$$

where $\lambda_0 + \lambda_1 + \cdots + \lambda_w = \lambda_1 + 2\lambda_2 + \cdots + w\lambda_w = w$. From (36) and (37), we get,

$$\{12\cdots w\} = (-1)^w \sum (-1)^{\lambda_1+\lambda_2+\cdots+\lambda_w} \frac{(\lambda_1+\lambda_2+\cdots+\lambda_w)!}{\lambda_1!\,\lambda_2!\cdots\lambda_w!} a_0^{\lambda_0} a_1^{\lambda_1} \cdots a_w^{\lambda_w}. \tag{38}$$

In MUIR's Determinants,‡ it is shown that

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* Cf. BURNSIDE and PANTON, *Theory of Equations*, second edition, p. 297.

† By the symmetry of (35), a_w is the same function of the t's, that t_w is of the a's, and we note two expressions for the a's similar to (36) and (37).

‡ Edition of 1882, p. 170, § 125.

$$(39)\qquad \frac{|i_{m+1}i_{m+2}\cdots i_{m+n}|}{|012\cdots n-1|}=\begin{vmatrix} t_{i_{m+1}} & t_{i_{m+2}} & \cdots & t_{i_{m+n}} \\ t_{i_{m+1}-1} & t_{i_{m+2}-1} & \cdots & t_{i_{m+n}-1} \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}=\{i_{m+1}i_{m+2}\cdots i_{m+n}\}_t,$$

where the meaning of the subscript t is obvious. Hence

$$\begin{vmatrix} i_1 i_2 \cdots i_m \\ p_1 p_2 \cdots p_m \end{vmatrix}=(-1)^w a_0^m \begin{Bmatrix} p_1 p_2 \cdots p_m \\ i_{m+1}\cdots i_{m+n} \end{Bmatrix}_t$$

or

$$(40)\qquad \begin{vmatrix} i \\ p \end{vmatrix}=(-1)^w a_0^m \begin{Bmatrix} p \\ \bar{i} \end{Bmatrix}_t.$$

We have, as an important inference from the preceding relations,

$$a_0^m\{i_{m+1}\cdots i_{m+n}\}_t=(-1)^w\{\bar{\imath}_1\bar{\imath}_2\cdots\bar{\imath}_m\}_a,$$

whence

$$(41)\qquad \begin{aligned} a_0^m\{i\}_t&=(-1)^w\{\bar{i}\}_a,\\ a_0^m\{\bar{i}\}_t&=(-1)^w\{i\}_a. \end{aligned}$$

This leads us to inquire whether, more generally,

$$a_0^m\{i_{m+1}-\lambda_1\cdots i_{m+n}-\lambda_1\}_t=(-1)^w\{i_1-\lambda_1\cdots i_m-\lambda_1\}_a.$$

By multiplying the right hand determinant by $\mu^{\sum_1^m(i_r-\lambda_r)}$, (the rth column by $\mu^{i_r-\lambda_1}$; the rth line, except the first, by $\mu^{\lambda_1-\lambda_r}$), we have, denoting the determinant by f,

$$(42)\qquad \mu^{\sum_1^m(i-\lambda)}f(a_0,a_1,\cdots a_m)=f(\mu^0 a_0,\mu^1 a_1,\cdots\mu^m a_m);$$

hence f is isobaric and of weight

$$\sum_{r=1}^{r=m}(i_r-\lambda_r).$$

Similarly the left hand determinant is isobaric and of weight

$$\sum_{r=m+1}^{r=m+n}i_r-\sum_{r=1}^{r=n}\lambda_r.$$

That equality may exist, it is at least necessary that

$$\sum_{r=1}^{r=m}(i_r-\lambda_r)=\sum_{r=m+1}^{r=m+n}i_r-\sum_{r=1}^{r=n}\lambda_r.$$

Since, as previously proved (l. c., p. 2),

$$\sum_{r=1}^{r=m} i_r - \frac{m(m-1)}{2} = \sum_{r=m+1}^{r=m+n} i_r - \frac{n(n-1)}{2},$$

this requires

$$(m+n-1)(m-n) = 2\sum_{n+1}^{m} \lambda_r, \tag{43}$$

which is not generally true, unless $n=m$, when it subsists, as $\lambda_{m+1} = \lambda_{m+2} = \cdots = 0$. But that the proposed relation is not generally true even when $n = m$, is seen by taking $m = n = 4$, $\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 5$, $\lambda_4 = 6$. Then, $(i = 2, 5, 6, 7)$,

$$a_0^4 \begin{vmatrix} t_1 & t_2 & t_4 & t_5 \\ t_0 & t_1 & t_3 & t_4 \\ 0 & 0 & t_1 & t_2 \\ 0 & 0 & t_0 & t_1 \end{vmatrix} = a_0^4 \begin{vmatrix} t_1 & t_2 \\ t_0 & t_1 \end{vmatrix}^2, \qquad \begin{vmatrix} a_0 & a_3 & a_4 & a_5 \\ 0 & a_2 & a_3 & a_4 \\ 0 & a_0 & a_1 & a_2 \\ 0 & 0 & a_0 & a_1 \end{vmatrix} = a_0 \begin{vmatrix} a_2 & a_3 & a_4 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \end{vmatrix}.$$

Therefore equality is impossible as the latter determinant contains a_4, while the former does not.*

§ 6. *A general expression for* $\left|\begin{smallmatrix} i \\ p \end{smallmatrix}\right|$.

Using (38) and (41) and putting the complex $p_1 p_2 \cdots p_m = 0^{x_0} 1^{x_1} 2^{x_2} \cdots m^{x_m}$, we have

$$\begin{vmatrix} 1\,2\cdots m \\ p_1 p_2 \cdots p_m \end{vmatrix} = \left\{ \begin{matrix} 0^{x_0} 1^{x_1} \cdots m^{x_m} \\ 1\,2 \cdots m \end{matrix} \right\} = (-1)^{m+x_1+x_2+\cdots+x_m} \frac{(x_1+x_2+\cdots+x_m)!}{x_1!\, x_2! \cdots x_m!}, \tag{44}$$

a singie explicit formula for the coefficient in the first column of our table.† It is also the summation of the coefficients in the symmetric function (p) excepting the factor $(-1)^m$, for by (32) and (44), we have

$$\sum_i \binom{i}{p} = (-1)^{x_1+x_2+\cdots+x_m} \frac{(x_1+x_2+\cdots+x_m)!}{x_1!\, x_2! \cdots x_m!}, \tag{45}$$

with $i_1 + i_2 + \cdots + i_m = m$.

By using the form (40), we can express other coefficients as sums of the form obtained for the first column, when $a_0^m \{ i_{m+1} \cdots i_{m+n} \}_t$ is expanded. If

* We note however the existence of the relation

$$\frac{|\bar{\imath}|}{|\bar{\lambda}|} = \frac{\{\bar{\imath}\}_t}{\{\bar{\lambda}\}_t} = \frac{\{i\}_a}{\{\bar{\lambda}\}_a},$$

where the indices of the lines of the determinants are consecutively decreased by unity, and where $\bar{\imath}$ and $\bar{\lambda}$ are of order n, and i and $\bar{\lambda}$ are of order m.

† See footnote to § 4 (32).

the last quarter of the table has been computed, we have this expansion.* In general, if $\left|\begin{smallmatrix} i \\ p \end{smallmatrix}\right|$ be among the first three-quarters of the coefficients of the table, and $|i_{m+1} \cdots i_{2m}|$ heads the conjugate column among the last quarter, then all coefficients in the same column with $\left|\begin{smallmatrix} i \\ p \end{smallmatrix}\right|$ can be found from the function,

$$(46) \qquad a_0^m \left\{ i_{m+1} \cdots i_{2m} \right\}_t = \sum \begin{vmatrix} i_{m+1} \cdots i_{2m} \\ q_1 \cdots q_m \end{vmatrix} t_{q_1} t_{q_2} \cdots t_{q_m},\dagger$$

and we have,

$$(47) \qquad \begin{vmatrix} i_1 i_2 \cdots i_m \\ p_1 p_2 \cdots p_m \end{vmatrix} = \sum_q (-1)^{m+r_1+r_2+\cdots} \begin{vmatrix} i_{m+1} \cdots i_{2m} \\ q_1 \cdots q_m \end{vmatrix} \sum_r \frac{r_1!\, r_2! \cdots}{\rho_1!\, \rho_2! \cdots \sigma_1!\, \sigma_2! \cdots},$$

where $i_{m+1} \cdots i_{2m}$ are the indices conjugate to $i_1 i_2 \cdots i_m$, and where $r_1, r_2, \cdots, \rho_1, \rho_2, \cdots, \sigma_1, \sigma_2, \cdots$, are defined as follows: $\epsilon_1 \epsilon_2 \cdots \epsilon_{r_1} \eta_1 \eta_2 \cdots \eta_{r_2} \cdots$ is any permutation of $p_1 p_2 \cdots p_m$ such that

$$\epsilon_1 + \epsilon_2 + \cdots + \epsilon_{r_1} = q_1, \qquad \rho_1 \text{ of the } \epsilon\text{'s are } 1, \qquad \rho_2 \text{ are } 2, \cdots,$$

$$\eta_1 + \eta_2 + \cdots + \eta_{r_2} = q_2, \qquad \sigma_1 \text{ of the } \eta\text{'s are } 1, \qquad \sigma_2 \text{ are } 2, \cdots,$$

. . . . . .

Thus we find for the second column, $(m > 1)$,

$$(48) \qquad \begin{vmatrix} 02 \cdots m-1\ m+1 \\ p_1 p_2 \cdots p_m \end{vmatrix} = (-1)^{m+1+x_1+x_2+\cdots+x_m} \frac{(x_2+\cdots+x_m)(x_1+\cdots+x_m-1)!}{x_1!\, x_2! \cdots x_m!},$$

for the third column, $(m > 3)$,

$$(49) \qquad \begin{vmatrix} 013 \cdots m-2\ m\ m+1 \\ p_1 p_2 \cdots p_m \end{vmatrix} = (-1)^{m+1+x_1+\cdots+x_m} \left(\frac{(x_1+\cdots+x_m-1)!}{(x_1-1)!\, x_2! \cdots x_m!} - \frac{(x_1+\cdots+x_m-2)!}{(x_1-2)!\, x_2! \cdots x_m!} - \frac{(x_1+\cdots+x_m-1)!}{x_1!\, (x_2-1)! \cdots x_m!} \right),$$

and for the fourth column, $(m > 5)$,

$$\begin{vmatrix} 0124 \cdots m-3\ m-1\ m\ m+1 \\ p_1 p_2 \cdots p_m \end{vmatrix}$$

$$(50) \qquad = (-1)^{m+1+x_1+\cdots+x_m} \left(\frac{(x_1+\cdots+x_m-2)!}{(x_1-2)!\, x_2! \cdots x_m!} + \frac{(x_1+\cdots+x_m-1)!}{x_1!\, (x_2-1)! \cdots x_m!} \right.$$

* The computation includes the self conjugate column or columns. In the last quarter of the table $\left|\begin{smallmatrix} i_{m-1} \cdots i_{2m} \\ q_1 \cdots q_m \end{smallmatrix}\right|$ is usually $+1$, -1, or 0, is practically a sign factor in most cases and its value is seen at once by writing down the notation, by means of § 3, 2.

† It will be seen that the t's play here an analogous rôle to that of the s's in the expression of a monomial symmetric function.

$$-\frac{(x_1+\cdots+x_m-3)!}{(x_1-3)!x_2!\cdots x_m!}-\frac{2(x_1+\cdots+x_m-2)!}{(x_1-1)!(x_2-1)!\cdots x_m!}-\frac{(x_1+\cdots+x_m-1)!}{x_1!x_2!(x_3-1)!\cdots x_m!}\Big).*$$

§ 7. *Constructed tables of coefficients.*

1. If $p_1 p_2 \cdots p_n$ is any partition of weight w, the partition $1^{p_1-p_2}2^{p_2-p_3}\cdots n^{p_n}$ is the conjugate partition of weight w, and it follows that the first is also the conjugate of the second partition. The complex $p_n, p_{n-1}+1, \cdots p_1+n-1$, and that one got by taking the numbers of the series $012\cdots m+n-1$, which remain after the complementaries of the former complex with respect to $m+n-1$, have been rejected, are obtained from the preceding conjugate partitions by adding $0, 1, 2, \cdots n-1$ and $0, 1, 2, \cdots m-1$ to their respective complexes arranged in ascending order of magnitude, and where $m \geqq p_1$. The first complex is evidently obtained as stated. The complementaries of its elements are the factors of the symbolical product of elements

$$\prod_1^n (m+r-p_r-1).$$

All other elements but these arranged in ascending order of magnitude are the $n+1$ factors of the product

$$\prod_0^n (m+r-p_r\cdots m+r-p_{r+1}-1),$$

where the $r+1$th factor is a product of p_r-p_{r+1} consecutive elements, and where $p_0=m$, $p_{n+1}=0$ and where any $r+1$th factor containing negative elements, or elements in descending order is to be rejected. If from the factors of this product we subtract the corresponding factors of $012\cdots m-1$, similarly expressed as

$$\prod_0^n (m-p_r\cdots m-p_{r+1}-1),$$

a product of $n+1$ factors in which the $(r+1)$th factor is the product of p_r-p_{r+1} consecutive elements, we have for the $(r+1)$th factor of the remainder $r^{p_r-p_{r+1}}$, and hence the remainder is

$$\prod_0^n r^{p_r-p_{r+1}} = \prod_1^n r^{p_r-p_{r+1}} = 1^{p_1-p_2}2^{p_2-p_3}\cdots n^{p_n}, \dagger \tag{51}$$

as was stated. On this account such complexes are called conjugate complexes, and alternants or determinants which have them for first line indices are called

* In these formulas terms containing negative factorials are to be placed equal to zero.

† The zero elements can be omitted as they merely call attention to the fact that $m-p_1$ places in the partition are not filled.

conjugate alternants or determinants. When $n = m$ they are called principal conjugate alternants or determinants. Every complex has one principal conjugate complex and an infinite number of other conjugate complexes. Analogous statements are true of alternants and determinants. Thus $|1234|$ and $|0127|$ are principal conjugate alternants, $\{1234\}$ and $\{0127\}$ principal conjugate determinants. Among the principal conjugates occur self-conjugate complexes, alternants or determinants. Thus $\{01347\}$ is a self-conjugate determinant.

2. For purposes of tabulation of coefficients, we write in a column at the left the different symmetric functions (partitions) of weight m, with conjugate partitions equidistant from the ends, and with the exception of the self-conjugates which will come in the middle, we arrange the upper portion in order from the least to the greatest distribution. That distribution is least, the number of elements being the same, which comes first in lexicographic order (e. g., 52 comes before 43), and the number of elements being different, that which has the least number being first (e. g., 43 comes before 51^2). It follows by considering the conjugates of two consecutive partitions, that the lower portion of the column will then be arranged, *so that either a partition with the same number of elements as the next comes first in lexicographic order, or one with fewer elements precedes, or if one with more elements comes first, it will contain a greater element than the following ones.* For of two consecutive partitions, the complexes are either of the form

$$(52)\qquad \begin{aligned} &a_1 a_2 \cdots a_\kappa a_{\kappa+1} \cdots a_r, \\ &a_1 a_2 \cdots a_\kappa - 1\ a_{\kappa+1} + 1 \cdots a_r, \end{aligned}$$

and the respective conjugates will occur in the order,

$$(53)\qquad \begin{aligned} &1^{a_1-a_2} 2^{a_2-a_3} \cdots (\kappa-1)^{a_{\kappa-1}+1-a_\kappa}\, \kappa^{a_\kappa-2-a_{\kappa+1}} (\kappa+1)^{a_{\kappa+1}+1-a_{\kappa+2}} \cdots \rho^{a_r}, \\ &1^{a_1-a_2} 2^{a_2-a_3} \cdots (\kappa-1)^{a_{\kappa-1}-a_\kappa}\, \kappa^{a_\kappa-a_{\kappa+1}} \cdots \rho^{a_r}, \end{aligned}$$

or they are of the form

$$(54)\qquad \begin{aligned} &a_1 a_2 \cdots a_r, \\ &a_1 + 1\, a_2 \cdots a_\kappa - 1 \cdots a_i - 1 \cdots a_{r+1}, \end{aligned}$$

and their conjugates will occur as

$$(55)\qquad \begin{aligned} &1^{a_1+1-a_2} 2^{a_2-a_3} \cdots \kappa^{a_\kappa-1-a_{\kappa+1}} \cdots i^{a_i-1-a_{i+1}} \cdots (r+1)^{a_{r+1}}, \\ &1^{a_1-a_2} 2^{a_2-a_3} \cdots \kappa^{a_\kappa-a_{\kappa+1}} \cdots i^{a_i-a_{i+1}} \cdots \rho^{a_r}. \end{aligned}$$

In the first case a conjugate partition with the same number (a_1) of elements lexicographically or with fewer elements (one less when $\kappa = 1$), in the second case one having a greater element $(r+1)$ precedes.

We next write in a horizontal line the same partitions, beginning at the bottom of the column and setting them down in the reverse column order from right to left. Above each one we place the complex obtained by adding $012\cdots m-1$ to its elements, zeros supplied to bring the number up to m, and all arranged in ascending order of magnitude before adding. These complexes give the indices of all the alternants possible in all of the products $A\phi$, where ϕ is any monomial symmetric function of weight m, and $A=|012\cdots m-1|$, or also the first line indices of all determinants possible in all products $a_0^m\phi$, with conjugate complexes equidistant from the ends. Since the different partitions are the diagonal indices of §3, 2, *the coefficient of any alternant which corresponds to a partition earlier than the function is zero*, because the lower complex will contain either fewer or greater elements, hence by the operation of §3, 2 it is impossible to make the lower equal to the upper complex; for, in the one case, while that operation can diminish, it can not increase the number of elements below, and in the other, while it can increase the element in the last place by passing it over preceding elements, it can not diminish it. *Thus we have generally proved, if* $q_1q_2\cdots q_m$ *precede* $p_1p_2\cdots p_m$,

$$\left[\begin{matrix}p_1p_2\cdots p_m\\ q_1q_2\cdots q_m\end{matrix}\right]=0. \tag{56}$$

The last coefficient which can have significance in a horizontal line is evidently the one

$$\left[\begin{matrix}p_1p_2\cdots p_m\\ p_1p_2\cdots p_m\end{matrix}\right]=1. \tag{57}$$

The formulas (56) *and* (57) *taken together show that our coefficients all lie in a triangle of the table, with the bounding diagonal of ones for its hypothenuse.**

3. *The sum of the coefficients in every column is zero, except for the last column which contains only unity.* Since $\left|\begin{smallmatrix}\kappa\\ p\end{smallmatrix}\right|=\left\{\begin{smallmatrix}p\\ \kappa\end{smallmatrix}\right\}$, the coefficients of the column headed by the complex κ, are the different coefficients in the development of $\{\kappa\}$. If the a's are put equal to unity $\{\kappa\}=0$, except for the last column, where $\{012\cdots m-2\,2m-1\}=1$; hence we have proved,

$$\sum_p\left|\begin{matrix}\kappa_1\kappa_2\cdots\kappa_m\\ p_1p_2\cdots p_m\end{matrix}\right|=0, \tag{58}$$

where the κ's are any complex of column indices, but the last, and the p's run through all partitions of weight m. This property affords a useful check in computing the tables. In accordance with the construction here explained and

* The triangle property would still hold for a table arranged in strict dictionary order, for then by §3, 2 $\left[\begin{smallmatrix}p\\ q\end{smallmatrix}\right]=0$, if q precedes p.

by the various methods of computing previously discussed, tables are given in this paper up to weight seven.

§ 8. *On the quotient of two alternants.*

The coefficients of any symmetric function in the quotient of an alternant $|\kappa|$ by the alternant $|012\cdots m-1|$, both being of order m, can be found as follows:

1. If the given alternant is in the table, let r be the number of removes of the symmetric function from the last line in which the alternant has a coefficient. Let s be the number of partitions in the table from this line to the end. Then the coefficient of the symmetric function in the quotient

$$\frac{|\kappa_1 \kappa_2 \cdots \kappa_m|}{|012\cdots m-1|} \quad \text{is} \quad (-1)^{s-r+\frac{s(s-1)}{2}}$$

times the determinant formed by taking the next $r-1$ lines ($r-i+1$ elements at the end from the ith of these lines) and for the last line r elements immediately above the r elements first chosen, with their signs changed. Thus the coefficient of (31^3) in the quotient,

$$\frac{|012369|}{|012345|} \text{ is } (-1)^{8-4+28} \begin{vmatrix} 1 & -1 & 0 & 1 \\ -1 & -1 & 1 & 0 \\ -2 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{vmatrix} = 3 \text{ (see § 4)}.$$

In fact if we multiply the different symmetric functions $\phi_1, \phi_2, \cdots$ involved by undetermined multipliers $\mu_1, \mu_2, \cdots$ and then so determine the μ's that

$$|012\cdots m-1| \sum \mu\phi = |\kappa|,$$

we shall obtain the result stated.

2. If the given alternant is not in a table, we take the difference

$$\kappa_1 \kappa_2 - 1 \kappa_3 - 2 \cdots \kappa_m - m + 1$$

of the complexes in the dividend and divisor and note the weight and partition of the complex, then we turn to the table of the given resulting weight, and take the column headed by the resulting partition and proceed exactly as in case 1. The explanation of this will appear in the next section. Thus the coefficient of (31^3) in $|012\cdots m\, m+3|/|012\cdots m-1|$ is 3.

§ 9. *On the invariance of the coefficients* $\left|\begin{smallmatrix} i \\ p \end{smallmatrix}\right|$.

In § 7 we have supposed the alternant $|012\cdots m-1|$ to be the multiplier of the symmetric functions of weight m. We will now show that *the whole*

product table for weight m remains absolutely invariant and still gives the complete product, if the column headings be merely altered suitably, when $|012\cdots m+r-1|$ is used as multiplier. In fact by § 3, 2,

$$(59)\qquad \begin{vmatrix} \kappa_1\kappa_2\cdots\kappa_{m+r} \\ 0^r p_1 p_2\cdots p_m \end{vmatrix} = \begin{bmatrix} 0^r p_1 p_2\cdots p_m \\ \kappa_1\kappa_2 - 1\cdots\kappa_{m+r} - m - r + 1 \end{bmatrix}.$$

To be significant the lower complex on the right hand side must be of weight m, must therefore contain at least r zeros $\kappa_1\ \kappa_2 - 1\cdots\kappa_r - r + 1$, and is some partition of weight m. Of such partitions however only those of the table exist, hence omitting zeros,

$$(60)\qquad \begin{vmatrix} \kappa_1\kappa_2\cdots\kappa_{m+r} \\ 0^r p_1 p_2\cdots p_m \end{vmatrix} = \begin{bmatrix} p_1 p_2\cdots p_m \\ \kappa_{r+1} - r\cdots\kappa_{m+r} - m - r + 1 \end{bmatrix},$$

where the right hand member of (60) is a coefficient of the table. By adding the elements of a partition of the horizontal line arranged in ascending order to the same number of elements at the end of $012\cdots m+r-1$, we obtain the new column heading. Thus the table for weight w is a table for an infinite number of products.*

If the multiplier is $|012\cdots m-r-1|$, then it is evident as before, that the table still gives the coefficients for those columns headed by a partition not containing more than $m-r$ elements, but that for the rest the multiplication will have to be indicated. From the invariance of the product table follows also the invariance of the quotient table.

Examples:

1. To find $|0123456789|(321^2)$.

We turn to the table of weight seven, and add 1^7, 21^5, 2^21^3, 2^31, 31^4, 321^2, to 0123456789 and obtain the complexes of the alternants in the product, and in the line 321^2, the respective coefficients and have,

$$|0123456789|(321^2) = -12|012456789\ 10| + 6|012356789\ 11|$$
$$(61)\qquad + |01234678\ 10\ 11| - 2|01234579\ 10\ 11| - 3|012346789\ 12|$$
$$+ |01234578\ 10\ 12|.$$

2. $$(62)\qquad |01|(4) = -|14| + |05| + \sum_3^m \alpha_s |01|.$$

*This invariance also appears from the standpoint that the table is also a table for symmetric functions. Using different alternants as multipliers in the manner indicated is equivalent, by the theorem of corresponding matrices, to multiplying the function by different powers of a_0, which can not affect the coefficients, since, as is well known, they are not functions of the powers of a_0.

The table also retains its invariance when the multiplier is $|\lambda, \lambda+1, \cdots m+r+\lambda-1|$, i. e., when each of the indices of $|012\cdots m+r-1|$ is increased by the same positive integer λ.

If we limit example 2 to a quadratic, $\alpha_3 = \alpha_4 = \cdots = \alpha_m = 0$, and the product is complete with the first two terms; this is another way of looking at this case.

§ 10. *The table as a table for symmetric functions.*

By (6), the second of the theorems mentioned in § 1, our table becomes a table for the expression of the symmetric function $a_0^m(p)$ as a sum of determinants of the a's of the mth order when conjugate column complexes are interchanged, and all the coefficients have been multiplied by $(-1)^w$.* As the coefficients in the expansion of the given determinants are already found in the columns headed by their first line complexes of indices, we have only to take these coefficients with their partitions at the left interpreted as a's, and multiply by the coefficient of the determinant and take the sum of all such determinants in order to get $a_0^m(p)$. Thus our table becomes a table for expressing (p) in terms of elementary products (cf. § 1).

Examples:

1. To find $a_0^2(2^2 1^2)$. From the table of weight six we find

$$a_0^6(2^2 1^2) = 6\{01234\ 11\} - 3\{01235\ 10\} + \{012369\}. \tag{63}$$

By the same table:

$$\{01234\ 11\} = a_0^5 a_6,$$

$$\{01235\ 10\} = -a_0^5 a_6 + a_0^4 a_1 a_5,$$

$$\{012369\} = -a_0^4 a_1 a_5 + a_0^4 a_2 a_4.$$

Introducing these values in (63) and dividing by a_0^4, we get

$$a_0^2 \sum \alpha_1^2 \alpha_2^2 \alpha_3 \alpha_4 = 9a_0 a_6 - 4a_1 a_5 + a_2 a_4. \tag{64}$$

2. To find (31^2). Put $a_0 = 1$; from the table of weight 5, by changing the signs of all the coefficients, since $(-1)^w = (-1)^5 = -1$, we have proceeding as before,

$$(31^2) = -3\{01239\} + \{01248\} + \{01257\} - \{01347\} \tag{65}$$

or

$$\sum \alpha_1^3 \alpha_2 \alpha_3 = -5a_5 + a_1 a_4 + 2a_2 a_3 - a_1^2 a_3. \tag{66}$$

* If the column complexes and the coefficients are used unchanged, then (p) is expressed as a sum of determinants in the t's, on account of the relations (41), § 5, and the expansion of these determinants is given by the table by interpreting the partitions at the left as t's.

§11. *Seven product tables.*

We give here product tables from weight one to weight seven inclusive.

WEIGHT 1.

		1 (1)
$\|0\|\times$	(1)	1

WEIGHT 2.

		12 (1^2)	03 (2)
$\|01\|\times$	(2)	−1	1
$a_0^2\times$	(1^2)	1	

WEIGHT 3.

		123 (1^3)	024 (21)	015 (3)
$\|012\|\times$	(3)	1	−1	1
$a_0^3\times$	(21)	−2	1	
	(1^3)	1		

WEIGHT 4.

		1234 (1^4)	0235 (21^2)	0145 (2^2)	0136 (31)	0127 (4)
	(4)	−1	1	0	−1	1
$\|0123\|\times$	(31)	2	−1	−1	1	
$a_0^4\times$	(2^2)	1	−1	1		
	(21^2)	−3	1			
	(1^4)	1				

WEIGHT 5.

		12345 (1^5)	02346 (21^3)	01356 (2^21)	01347 (31^2)	01257 (32)	01248 (41)	01239 (5)
	(5)	1	−1	0	1	0	−1	1
	(41)	−2	1	1	−1	−1	1	
$\|01234\|\times$	(32)	−2	2	−1	−1	1		
$a_0^5\times$	(31^2)	3	−1	−1	1			
	(2^21)	3	−2	1				
	(21^3)	−4	1					
	(1^5)	1						

WEIGHT 6.

		123456 (1^6)	023457 (21^4)	013467 (2^21^2)	012567 (2^3)	013458 (31^3)	012468 (321)	012459 (41^2)	012378 (3^2)	012369 (42)	012356 10 (51)	012347 11 (6)
	(6)	−1	1	0	0	−1	0	1	0	0	−1	1
	(51)	2	−1	−1	0	1	1	−1	0	−1	1	
	(42)	2	−2	1	−1	1	0	−1	−1	1		
$\lvert 012345 \rvert \times$	(3^2)	1	−1	0	1	1	−1	0	1			
$a_0^6 \times$	(41^2)	−3	1	1	1	−1	−1	1				
	(321)	−6	4	0	−2	−2	1					
	(31^3)	4	−1	−1	0	1						
	(2^3)	−1	1	−1	1							
	(2^21^2)	6	−3	1								
	(21^4)	−5	1									
	(1^6)	1										

WEIGHT 7.

		1234567 (1^7)	0234568 (21^5)	0134578 (2^21^3)	0124678 (2^31)	0134569 (31^4)	0124579 (321^2)	0123679 (32^2)	0124568 10 (41^3)	0123589 (3^21)	0123457 10 (421)	0123456 11 (51^2)	0123489 10 (43)	0123478 11 (52)	0123467 12 (61)	0123458 13 (7)
	(7)	1	−1	0	0	1	0	0	−1	0	0	1	0	0	−1	1
	(61)	−2	1	1	0	−1	−1	0	1	0	1	−1	0	−1	1	
	(52)	−2	2	−1	1	−1	0	−1	1	1	0	−1	−1	1		
	(43)	−2	2	0	−1	−2	1	1	1	−1	−1	0	1			
	(51^2)	3	−1	−1	−1	1	1	1	−1	0	−1	1				
$\lvert 0123456 \rvert \times$	(421)	6	−4	0	0	2	1	−1	−2	−1	1					
$a_0^7 \times$	(3^21)	3	−2	−1	2	2	−1	−1	0	1						
	(41^3)	−4	1	1	1	−1	−1	0	1							
	(32^2)	3	−3	2	−1	1	−1	1								
	(321^2)	−12	6	1	−2	−3	1									
	(31^4)	5	−1	−1	0	1										
	(2^31)	−4	3	−2	1											
	(2^21^3)	10	−4	1												
	(21^5)	−6	1													
	(1^7)	1														

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THE GROUPS OF ORDER p^3q^β*

BY

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§ 1. *Introduction.*

The researches of FROBENIUS† and BURNSIDE,‡ familiar to all students of the theory of groups, have established the non-existence of simple groups of orders pq^β and p^2q^β, p and q being different primes, and the consequent solvability of all groups of these orders. In the following paper I show that all groups of order p^3q^β are compound, and therefore also solvable. For convenience of reference I place at the beginning of the discussion the two following theorems of which repeated use is made in the subsequent reasoning:

I. *A simple group $\mathfrak{G}$ of order $g = p^\alpha q^\beta$, p and q being different primes and $p > q$, cannot contain any subgroup $\mathfrak{H}$ of order h whose index g/h in $\mathfrak{G}$ is $< p^2$.*

In view of the results of FROBENIUS and BURNSIDE cited above, we assume that $\alpha > 2$.

The case $g/h < p$ is trivial. Here every group $\mathfrak{A}$ of order p^α in $\mathfrak{G}$ transforms every conjugate of $\mathfrak{H}$ into itself and $\mathfrak{G}$ is certainly compound. Again, for $g/h = p$, $h = p^{\alpha-1}q^\beta$, every subgroup of order $p^{\alpha-1}$ in $\mathfrak{H}$ transforms $\mathfrak{H}$ and therefore every conjugate of $\mathfrak{H}$ into itself, and $\mathfrak{G}$ is again certainly compound.

Suppose, now, that $\mathfrak{K}$ is the largest subgroup of $\mathfrak{G}$ that contains $\mathfrak{H}$ and is less that $\mathfrak{G}$. Since $\mathfrak{G}$ is simple, $\mathfrak{K}$ is invariant under only those elements of $\mathfrak{G}$ which are contained in $\mathfrak{K}$. The index g/k of $\mathfrak{K}$ in $\mathfrak{G}$ is $\leqq g/h$ and $> p$; $\mathfrak{K}$ has g/k distinct conjugates in $\mathfrak{G}$.

1) Let $g/k = pq^i (0 < q^i < p)$, $k = p^{\alpha-1}q^{\beta-i}$. Having less than p^2 conjugates in $\mathfrak{G}$, $\mathfrak{K}$ contains a subgroup of order $p^{\alpha-1}$ from every group $\mathfrak{A}$ of order p^α in $\mathfrak{G}$. Every subgroup of order $p^{\alpha-1}$ of $\mathfrak{K}$ is permutable with every subgroup $\mathfrak{L}$ of order $q^{\beta-i}$ of $\mathfrak{K}$. $\mathfrak{L}$ is invariant under a subgroup of $\mathfrak{G}$ of order $g^{\beta-i+1}$, and this transforms $\mathfrak{K}$ into at least one conjugate $\mathfrak{K}'$ of $\mathfrak{K}$, different from $\mathfrak{K}$.

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† FROBENIUS: Berliner Sitzungsberichte, 1895, p. 185; cf. Acta Mathematica, vol. 26 (1902), p. 198.

‡ BURNSIDE: *Theory of Groups*, p. 348. Cf. JORDAN, Liouville's Journal, ser. 5, vol. 4 (1898), pp. 21–26.

$\mathfrak{K}$ and $\mathfrak{K}'$, having $\mathfrak{L}$ in common, have no group of order $p^{\alpha-1}$ in common. Then $\mathfrak{L}$ is permutable with two different groups of order $p^{\alpha-1}$ contained in the same group $\mathfrak{A}$ of order p^α, and is therefore permutable with $\mathfrak{A}$. $\mathfrak{L}$ and $\mathfrak{A}$ generate a group of order $p^\alpha q^{\beta-i}$ contained in $\mathfrak{G}$ and having $< p$ conjugates in $\mathfrak{G}$.

2) Let $g/k = q^j (p < q^j < p^2)$, $k = p^\alpha q^{\beta-j}$. As in 1), every conjugate of $\mathfrak{K}$ contains a subgroup of order $p^{\alpha-1}$ from every group $\mathfrak{A}$ of order p^α in $\mathfrak{G}$, and every subgroup $\mathfrak{L}$ of order $q^{\beta-j}$ of $\mathfrak{K}$ occurs also in a conjugate $\mathfrak{K}'$ of $\mathfrak{K}$ different from $\mathfrak{K}$. $\mathfrak{K}$ and $\mathfrak{K}'$ have in common the group $\mathfrak{L}$ and a group of order $p^{\alpha-1}$ from every group $\mathfrak{A}$ of $\mathfrak{K}$ or $\mathfrak{K}'$; their greatest common subgroup $\mathfrak{D}$ is of order $p^{\alpha-1}q^{\beta-j}$. All the conjugates of $\mathfrak{D}$ in $\mathfrak{K}$ or $\mathfrak{K}'$ are obtained by transforming $\mathfrak{D}$ by any group $\mathfrak{A}$ of $\mathfrak{K}$ or $\mathfrak{K}'$. Hence all the subgroups of order $p^{\alpha-1}$ in $\mathfrak{D}$ are common to all the conjugates of $\mathfrak{D}$ in $\mathfrak{K}$ and $\mathfrak{K}'$. The subgroups of order $p^{\alpha-1}$ of $\mathfrak{D}$ generate a group invariant in $\mathfrak{K}$ and in $\mathfrak{K}'$, and therefore in a group $\mathfrak{M}$ contained in $\mathfrak{G}$ and containing $\mathfrak{K}$ and $> \mathfrak{K}$. Then $\mathfrak{M} = \mathfrak{G}$, and $\mathfrak{G}$ is compound.

II. *If a simple group $\mathfrak{G}$ of order $g = p^\alpha q^\beta$, p and q being, as in* I, *different primes and $p > q$, contains a subgroup $\mathfrak{H}$ of order $p^i q^j$ where $1 < q^{\beta-j} < p$, then $\mathfrak{H}$ is contained in a subgroup $\mathfrak{K}$ of $\mathfrak{G}$ of order $p^{i+x}q^\beta$ $(x \geqq 0, i + x < \alpha)$.*

Suppose that the largest group containing $\mathfrak{H}$ and contained in $\mathfrak{G}$ and $< \mathfrak{G}$ is $\mathfrak{K}$ of order $k = p^{i+x}q^{j+y} (j + y < \beta)$; $\mathfrak{K}$ is invariant under only those elements of $\mathfrak{G}$ that are contained in $\mathfrak{K}$; having $< p^{\alpha-i-x+1}$ conjugates in $\mathfrak{G}$, $\mathfrak{K}$ contains a subgroup of order p^{i+x} from every group $\mathfrak{A}$ of order p^α in $\mathfrak{G}$. If $i + x = \alpha$, $\mathfrak{G}$ is compound, by I. If $i + x < \alpha$ and $j + y < \beta$, a subgroup $\mathfrak{L}$ of order q^{j+y} of $\mathfrak{K}$ occurs in a conjugate $\mathfrak{K}'$ of $\mathfrak{K}$ different from $\mathfrak{K}$. $\mathfrak{L}$ is permutable with two groups of order p^{i+x} from the same group $\mathfrak{A}$; these with $\mathfrak{L}$ generate a group $\mathfrak{M}$ of order $p^{i+x+z}q^{j+y}$ $(z > 0)$ containing $\mathfrak{K}$ and contained in $\mathfrak{G}$ and $< \mathfrak{G}$; but this is contrary to assumption.

A simple application of Theorem II is afforded by the groups of order p^2q^β $(p > q > 2)$. A simple group of this order must contain p^2 subgroups $\mathfrak{B}$ of order q^β, p of which have a common subgroup $\mathfrak{D}$ of order $q^r (r > 0)$ invariant in a group $\mathfrak{D}'$ of order pq^{r+s} $(s > 0)$; $p^2 - 1$ is divisible by $q^{\beta-r}$, and $p - 1$ by q^s, hence $p - 1$ is divisible by $q^{\beta-r}$; $\mathfrak{D}'$ is contained in a subgroup of order pq^β of $\mathfrak{G}$. But then $\mathfrak{G}$ is compound. The theorem controls also the case $q = 2$, except in the single event that $s = 1$ and $p + 1 = 2^{\beta-r-1}$.

§ 2. *Preliminary treatment of the groups of order p^3q^β.*

A simple group of order $p^\alpha q^\beta$ $(p \leqq q)$ can occur only if $\alpha > 2\mu$, μ being the lowest index for which $p^\mu \equiv 1 \pmod q$.* For $\alpha = 3$, we can only take $\mu = 1$. A group $\mathfrak{G}$ of order p^3q^β can be simple only if $p \equiv 1 \pmod q$; also $\beta > 2\nu$, where $q^\nu \equiv 1 \,(p)$, therefore $q^\beta > p^2$.

* BURNSIDE, *Theory of Groups*, p. 345. Cf. FROBENIUS, Acta Mathematica, vol. 26 (1902), p. 194.

A simple group $\mathfrak{G}$ of order p^3q^β must contain either p^2 or p^3 subgroups $\mathfrak{B}$ of order q^β. Since $q^\beta > p^2$, the elements of these subgroups $\mathfrak{B}$ cannot be wholly distinct in either case.

If $\mathfrak{G}$ contains only p^2 subgroups $\mathfrak{B}$, and if two of these are so chosen that the order q^r of their greatest common divisor $\mathfrak{D}$ is a maximum, then $\mathfrak{D}$ is invariant in a subgroup $\mathfrak{D}'$ of $\mathfrak{G}$ whose order is $p^x q^{r+s}$ $(x = 1, 2; s > 0)$ and which contains exactly p groups of order q^{r+s}. Here $p^2 - 1$ is divisible by $q^{\beta-r}$, therefore $p + 1 \geqq q^{\beta-r-1}$. (If $q \neq 2$, $p - 1$ is divisible by $q^{\beta-r}$, and $\mathfrak{D}'$ has less than p^2 conjugates in $\mathfrak{G}$ unless $x = 1$. For odd q, the discussion can be greatly simplified, as in the case of order p^2q^β). Each of the $p^{3-x}q^{\beta-r-s}$ conjugate groups $\mathfrak{D}$ occurs in exactly p of the groups $\mathfrak{B}$. If each of the p^2 groups $\mathfrak{B}$ contains k of the groups $\mathfrak{D}$, the total number of the groups $\mathfrak{D}$ is $p^2k/p = p^{3-x}q^{\beta-r-s}$; hence $k = p^{2-x}q^{\beta-r-s}$. If now $x = 1$, each group $\mathfrak{B}$ contains $pq^{\beta-r-s}$ groups $\mathfrak{D}$; each of the latter is contained in $p - 1$ other groups $\mathfrak{B}$ and no two of them occur together in any second group $\mathfrak{B}$. But there are only p^2 of the groups $\mathfrak{B}$ and $p^2 < pq^{\beta-r-s}(p-1) + 1$, unless $q^{\beta-r-s} = 1$, $r + s = \beta$. Then $\mathfrak{D}'$ is of order pq^β, has exactly p^2 different conjugates in $\mathfrak{G}$, and therefore contains an element P of order p from every subgroup $\mathfrak{A}$ of order p^3 in $\mathfrak{G}$.

Let $\mathfrak{A}$ be any subgroup of order p^3 in $\mathfrak{G}$, and let $\mathfrak{B}$ occur in $\mathfrak{D}'$; having only p^2 conjugates in $\mathfrak{G}$, $\mathfrak{B}$ is invariant under an element P of order p in $\mathfrak{A}$; $\mathfrak{B}$ is also permutable with a subgroup $\mathfrak{A}_1$ of order p in $\mathfrak{A}$ not containing P. P transforms $\mathfrak{D}'$ into a conjugate of $\mathfrak{D}'$ different from $\mathfrak{D}'$ and containing the group $P^{-1}\mathfrak{A}_1 P$, which is different from $\mathfrak{A}_1$ but is contained with $\mathfrak{A}_1$ in a subgroup $\mathfrak{A}_2$ of order p^2 of $\mathfrak{A}$. $\mathfrak{B}$ is permutable with both $\mathfrak{A}_1$ and $P^{-1}\mathfrak{A}_1 P$ and therefore with $\mathfrak{A}_2$; $\mathfrak{A}_2$ and $\mathfrak{B}$ generate a group of order p^2q^β contained in $\mathfrak{G}$ and having only p conjugates in $\mathfrak{G}$.

Again, if $x = 2$ each group $\mathfrak{B}$ contains $q^{\beta-r-s}$ of the groups $\mathfrak{D}$. The p groups $\mathfrak{B}$ which have subgroups of order q^{r+s} in a same group $\mathfrak{D}'$ contain $p(q^{\beta-r-s} - 1) + 1$ of the groups $\mathfrak{D}$, and these are transformed among themselves by every element of order p in $\mathfrak{D}'$. All the elements of order p in $\mathfrak{D}'$ are therefore permutable with each of the remaining $p - 1$ groups $\mathfrak{D}$; they generate a group which is invariant in p groups $\mathfrak{D}'$ and therefore in a group $\mathfrak{M}$ of order $p^{2+y}q^{r+s+t}$ $(y = 0, 1)$ contained in $\mathfrak{G}$. If $y = 1$, $\mathfrak{M}$ has at most $p + 1$ conjugates in $\mathfrak{G}$. And if $y = 0$, $t > 0$ and $\mathfrak{M}$ has at most $p(p+1)/q < p^2$ conjugates in $\mathfrak{G}$.

$\mathfrak{G}$ must therefore contain p^3 sub-groups $\mathfrak{B}$. The maximum greatest common divisor $\mathfrak{D}$, of order q^r, of two of these is again invariant in a subgroup $\mathfrak{D}'$ of order $p^x q^{r+s}$ $(x = 1, 2)$ of $\mathfrak{G}$. Here $p^3 - 1$ is divisible by $q^{\beta-r}$, and $p - 1$, being divisible by q, is divisible by $q^{\beta-r-1}$ (in fact by $q^{\beta-r}$ if $q \neq 3$). Then, by the reasoning employed in the proof of Theorem II, if $r + s < \beta$, $\mathfrak{D}'$ is contained in a sub-group of order p^2q^β of $\mathfrak{G}$. Hence $r + s = \beta$ and $\mathfrak{D}'$ is of order

pq^β. Each group $\mathfrak{D}$ is common to p groups $\mathfrak{B}$. If each group $\mathfrak{B}$ contains k groups $\mathfrak{D}$, we have $p^3k/p = p^2$, hence $k = 1$; each group $\mathfrak{D}'$ contains precisely one group $\mathfrak{D}$. Each group $\mathfrak{B}$ occurs in only one group $\mathfrak{D}'$.

§ 3. *Final Investigation of the Groups of Order* p^3q^β.

Each of the p^3 groups $\mathfrak{B}$ of $\mathfrak{G}$ transforms among themselves the $p^3 - p$ conjugates of $\mathfrak{B}$ not contained in the group $\mathfrak{D}'$ in which $\mathfrak{B}$ occurs. Let $\mathfrak{D}'$, and $\mathfrak{B}$ in $\mathfrak{D}'$, be so chosen that a subgroup Δ common to $\mathfrak{B}$ and a conjugate of $\mathfrak{B}$ not contained in $\mathfrak{D}'$ is of the largest possible order, and let this order be q^ρ. Then $q^{\beta-\rho}$ divides $p^3 - p$, and therefore divides $p^2 - 1$; $\rho > 0$, and in general $\rho > q^{\beta-\rho-1}$, the only exception occurring when $q = 2$ and $p + 1 = 2^{\beta-\rho-1}$. In this exceptional case $\beta - r = 1$, $\mathfrak{D}$ is of order $2^{\beta-1}$.

The group Δ is common to two groups $\mathfrak{B}$ from different groups $\mathfrak{D}'$. Δ is invariant under subgroups $\mathfrak{R}_1$, $\mathfrak{R}_2$ of order $q^{\rho+\sigma_1}$, $q^{\rho+\sigma_2}$ $(\sigma_1, \sigma_2 > 0)$ of these two groups $\mathfrak{B}$. $\mathfrak{R}_1$ and $\mathfrak{R}_2$ cannot be contained in any subgroup of $\mathfrak{G}$ of order $q^{\rho+\tau}$ $(\tau > 0)$, for this subgroup would be common to two groups $\mathfrak{D}'$ and therefore to two groups $\mathfrak{B}$ contained one in each of these two groups $\mathfrak{D}'$. Δ is invariant in a subgroup Δ' of order $p^x q^{\rho+\sigma}$ $(x, \sigma > 0)$ of $\mathfrak{G}$. Any subgroup of order p^x of Δ' transforms $\mathfrak{D}'$ containing Δ into precisely p groups $\mathfrak{D}'$ each containing Δ, for Δ cannot occur in all the p^2 groups $\mathfrak{D}'$. Δ' has one or more subgroups of order $q^{\rho+\sigma}$ common with each of these p groups $\mathfrak{D}'$, and no subgroup of order $q^{\rho+\tau}$ $(\tau > 0)$ common with any other group $\mathfrak{D}'$. Hence Δ occurs in precisely p groups $\mathfrak{D}'$.

1) If $\rho > q^{\beta-\rho-1}$, or if $\sigma > 1$, then by Theorem II, Δ' is contained in a subgroup $\mathfrak{M}$ of order pq^β of $\mathfrak{G}$ (the order p^2q^β being inadmissible). $\mathfrak{M}$ contains p groups $\mathfrak{B}$ having Δ as their common subgroup; Δ is invariant in $\mathfrak{M}$, $\mathfrak{M} = \Delta'$, $\rho + \sigma = \beta$; and Δ has p^2 conjugates in $\mathfrak{G}$. If each group $\mathfrak{D}'$ contains k groups Δ, then since each group Δ occurs in p groups $\mathfrak{D}'$ we have $p^2k/p = p^2$, hence $k = p$.

If now two groups $\mathfrak{D}'$ have more than one group Δ in common, their greatest common divisor $\mathfrak{C}$ is of order pq^ρ and contains p groups Δ; these are all the groups Δ contained in the two groups $\mathfrak{D}'$; $\mathfrak{C}$ is the smallest group that contains them; $\mathfrak{C}$ is invariant in both groups $\mathfrak{D}'$ and has $< p^2$ conjugates is $\mathfrak{G}$.

If no two groups $\mathfrak{D}'$ have more than one group Δ in common, the p groups Δ contained in any group $\mathfrak{D}'$ are distributed among $p(p-1)+1$ groups $\mathfrak{D}'$, and none of them occur in $p - 1$ groups $\mathfrak{D}'$. All the elements of order p of a group $\mathfrak{D}'$ are therefore permutable with each of $p - 1$ other groups $\mathfrak{D}'$; these elements of order p are common to p groups $\mathfrak{D}'$ and are all the elements of order p of any of these p groups $\mathfrak{D}'$; they generate a group invariant under the p groups $\mathfrak{D}'$ and having $< p^2$ conjugates in $\mathfrak{G}$.

2) It remains to consider the special case $q = 2$, $p + 1 = 2^{\beta-\rho-1}$, $\sigma = 1$, $r = \beta - 1$. Let $\mathfrak{D}'$, $\mathfrak{D}'_1$ be two groups $\mathfrak{D}'$ having a group Δ in common. Δ

cannot be the greatest common divisor of $\mathfrak{D}_1'$ and $\mathfrak{D}_2'$, since either of the latter would then transform the other into $p2^{\beta-\rho} > p^2$ conjugates. The greatest common divisor $\mathfrak{C}$ of $\mathfrak{D}_1'$, $\mathfrak{D}_2'$ is therefore of order $p2^\rho$.

Suppose first that $\mathfrak{C}$ contains only one group of order 2^ρ, that is, that Δ is invariant in $\mathfrak{C}$; then Δ' contains $\mathfrak{C}$. If Δ' were of order $p2^{\rho+1}$, $\mathfrak{C}$ would be invariant in Δ' and would contain every element of order p of Δ'; whereas Δ' must contain an element of order p or p^2 which transforms $\mathfrak{D}_1'$ into $\mathfrak{D}_2'$. Hence Δ' is of order $p^2 2^{\rho+1}$, and Δ has $p2^{\beta-\rho-1} = p(p+1)$ conjugates in $\mathfrak{G}$. If each group $\mathfrak{D}'$ contains k groups Δ, we have $p^2k/p = p(p+1), k = p+1$. Since Δ is invariant under an element p of $\mathfrak{D}'$, Δ is contained in the group $\mathfrak{D}$ occurring in $\mathfrak{D}'$; the $p+1$ groups Δ occurring in $\mathfrak{D}'$ are conjugate in $\mathfrak{D}'$ and are therefore all contained in $\mathfrak{D}$. No two groups Δ occurring in the same group $\mathfrak{D}'$ can occur together in any second group $\mathfrak{D}'$. The p groups $\mathfrak{D}'$ which have Δ in common, contain p^2+1 groups Δ, and do not contain any one of the remaining $p-1$ groups Δ. Δ' transforms among themselves the p^2+1 groups Δ occurring with Δ in groups $\mathfrak{D}'$, and transforms among themselves the remaining $p-1$ groups Δ. All the elements of order p or p^2 in Δ' are permutable with these $p-1$ groups Δ; they all occur in p groups Δ' and are all the elements of order p or p^2 of each of these groups Δ'; they generate a group invariant under p groups Δ' and having $< p^2$ conjugates in $\mathfrak{G}$.

The group $\mathfrak{C}$ common to any two groups $\mathfrak{D}_1'$ and $\mathfrak{D}_2'$ must therefore contain p groups Δ. No group Δ is contained in a group $\mathfrak{D}$, for then Δ would be invariant in $\mathfrak{C}$. Since $\mathfrak{D}$ is of order $2^{\beta-1}$, Δ has a subgroup $\mathfrak{E}$ of order $2^{\rho-1}$ common with $\mathfrak{D}$. $\mathfrak{E}$ is invariant in $\mathfrak{C}$, since $\mathfrak{C}$ cannot have two groups of order $2^{\rho-1}$ common with $\mathfrak{D}$. $\mathfrak{E}$ is also invariant under subgroups of order $2^{\rho+\tau_1}$, $2^{\rho+\tau_2}$ $(\tau_1, \tau_2 \geqq 0)$ of $\mathfrak{D}_1$ and $\mathfrak{D}_2$, and therefore under subgroups $\mathfrak{L}_1$, $\mathfrak{L}_2$ of order $p2^{\rho+\tau_1}$, $p2^{\rho+\tau_2}(\tau_1, \tau_2 > 0)$ of $\mathfrak{D}_1'$ and $\mathfrak{D}_2'$ respectively. $\mathfrak{L}_1$ and $\mathfrak{L}_2$ cannot both be contained in a group of order $p2^{\rho+\tau}$ containing $\mathfrak{C}$. The largest group $\mathfrak{E}'$ contained in $\mathfrak{G}$ and containing $\mathfrak{E}$ as invariant subgroup is therefore of order $p^2 2^{\rho+\tau}$, where we must take $\tau = 1$, by virtue of Theorem II. $\mathfrak{E}'$ transforms $\mathfrak{D}'$ containing $\mathfrak{E}$ into p groups $\mathfrak{D}'$ containing $\mathfrak{E}$ and has $p2^{\rho+1}$ elements in common with each of these p groups $\mathfrak{D}'$. $\mathfrak{E}$ does not occur in any other group $\mathfrak{D}_i'$. For any group of order p^2 in $\mathfrak{E}'$ could transform $\mathfrak{D}_i'$ into only p groups $\mathfrak{D}'$; $\mathfrak{E}'$ would have an element P of order p common with $\mathfrak{D}_i'$; $\mathfrak{E}$, being invariant under P, would be contained in $\mathfrak{D}_i$ and would be invariant under a group of order 2^ρ of $\mathfrak{D}_i$; this group of order 2^ρ would occur in $\mathfrak{E}'$ and therefore in one of the p groups $\mathfrak{D}'$ into which $\mathfrak{E}'$ transforms $\mathfrak{D}_1'$; and this would lead to the case already disposed of where $\mathfrak{C}$ contains only one group Δ.

The group $\mathfrak{E}$ has $p(p+1)$ conjugates in $\mathfrak{G}$ and $p+1$ conjugates in each group $\mathfrak{D}'$. The $p(p+1)$ groups $\mathfrak{E}'$ are all different. The $p+1$ conjugates of $\mathfrak{E}$ which occur in $\mathfrak{D}'$ are conjugate in $\mathfrak{D}'$ and are all contained in $\mathfrak{D}$. No

two groups $\mathfrak{D}'$ can have two groups $\mathfrak{S}$ in common, since this would again lead to the rejected case where $\mathfrak{C}$ contains only one group Δ. The p groups $\mathfrak{D}'$ which have $\mathfrak{S}$ in common contains p^2+1 groups $\mathfrak{S}$. $\mathfrak{S}'$ transforms these p^2+1 groups $\mathfrak{S}$ among themselves. All the elements of order p or p^2 in $\mathfrak{S}'$ are permutable with each of the remaining $p-1$ groups $\mathfrak{S}$; they generate a group invariant in p groups $\mathfrak{S}'$ and having $< p^2$ conjugates in $\mathfrak{G}$.

COLUMBIA UNIVERSITY,
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GREEN'S THEOREM AND GREEN'S FUNCTIONS FOR CERTAIN SYSTEMS OF DIFFERENTIAL EQUATIONS*

BY

MAX MASON

The Green's function, defined originally for the potential equation, has been generalized to apply to the linear differential equation of second order of the elliptic type.† In the present paper the further generalization of Green's theorem and function to certain systems of differential equations will be made, and, the existence of the Green's functions being postulated, theorems regarding them will be established.

§ 1. *Green's Theorem.*

Consider the system of differential expressions

$$(1)\quad L_i(u_1, u_2, \cdots, u_n) = \Delta u_i + \sum_{j=1}^{n} \left\{ a_{ij} \frac{\partial u_j}{\partial x} + b_{ij} \frac{\partial u_j}{\partial y} + c_{ij} u_j \right\} \quad (i = 1, 2, \cdots, n),$$

where Δ is the Laplace operator; a_{ij}, b_{ij}, c_{ij} are continuous functions of x, y; and a_{ij}, b_{ij} have continuous first derivatives.

Let u_i, v_i be any functions of x, y which, within a region Ω and on its boundary, possess at every point second derivatives with respect to x and second derivatives with respect to y. The use of the ordinary formulae for integration by parts gives, after rearrangement of terms, the following *Green's theorem for the system* (1):

$$\sum_{i=1}^{n} \int\int_{(\Omega)} \{ v_i L_i(u_1, \cdots, u_n) - u_i M_i(v_1, \cdots, v_n) \} dx\, dy = \int_{(S)} (Q dy - P dx),$$

where S is the boundary of the region Ω, and

$$M_i(v_1, \cdots, v_n) = \Delta v_i - \sum_{j=1}^{n} \left\{ \frac{\partial (a_{ji} v_j)}{\partial x} + \frac{\partial (a_{ji} v_j)}{\partial y} - c_{ji} v_j \right\},$$

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† See Encyklopädie der mathematischen Wissenschaften, IIA7c (SOMMERFELD), § 3.

$$P = \sum_{i=1}^{n} \left\{ v_i \frac{\partial u_i}{\partial y} - u_i \frac{\partial v_i}{\partial y} + v_i \sum_{j=1}^{n} b_{ij} u_j \right\},$$

$$Q = \sum_{i=1}^{n} \left\{ v_i \frac{\partial u_i}{\partial x} - u_i \frac{\partial v_i}{\partial x} + v_i \sum_{j=1}^{n} a_{ij} u_j \right\}.$$

Applying the formulae:

$$\frac{dy}{ds} = \frac{dx}{dn} = \cos(n, x),$$

$$\frac{dx}{ds} = -\frac{dy}{dn} = -\cos(n, y),$$

where s is the length of arc along S and n the outward drawn normal, the following form is deduced:

$$(2)\quad \sum_{i=1}^{n} \int\int_{(\Omega)} \{ v_i L_i(u_1, \cdots, u_n) - u_i M_i(v_1, \cdots, v_n) \}\, dx\, dy$$

$$= \sum_{i=1}^{n} \int_{(S)} \left\{ v_i \frac{\partial u_i}{\partial n} - u_i \frac{\partial v_i}{\partial n} + v_i \sum_{j=1}^{n} [a_{ij} \cos(n, x) + b_{ij} \cos(n, y)] u_j \right\} ds.$$

The system of differential expressions $M_i(v_1, \cdots, v_n)$ will be called the *adjoint system* to $L_i(u_1, \cdots, u_n)$.* It follows that the relation is reciprocal, the system L_i being adjoint to M_i. The conditions that the system L_i be *self adjoint* are easily seen to be:

$$a_{ij} = -a_{ji}, \qquad b_{ij} = -b_{ji},$$

$$\frac{\partial a_{ji}}{\partial x} + \frac{\partial b_{ji}}{\partial y} = c_{ji} - c_{ij} \qquad (i, j = 1, 2, \cdots, n).$$

§ 2. *The Unique Determination of Solutions by their Boundary Values.*

As an application of (2) let us consider under what conditions a set of solutions† of the system

$$(3)\quad L_i(u_1, \cdots, u_n) = f_i \qquad (i = 1, 2, \cdots, n)$$

within a region Ω, where the f_i are given continuous functions of x, y, are uniquely determined by their boundary values. This is equivalent to asking under what conditions a set of solutions, not all zero, of the homogeneous system

* Professor BÔCHER, to whom I am indebted for many valuable suggestions and criticisms, has called my attention to the introduction by JORDAN of adjoint systems of ordinary differential equations of the first order. JORDAN, *Cours d'analyse*, vol. 3, p. 144.

† By a "set of solutions" within Ω we shall mean a set of functions which satisfy the equations at every point of Ω, thus demanding the existence of the second derivatives and continuity of the first.

$$L_i(u_1, \cdots, u_n) = 0, \tag{4}$$

which vanish on the boundary, can not exist.

Let the functions u_i of (2) be a set of solutions of (4) vanishing on S, and choose $v_i = u_i$. Then from (2):

$$-\sum_{i=1}^{n} \int\int_{(\Omega)} u_i M_i(u_1, \cdots, u_n)\, dx\, dy = \sum_{i=1}^{n} \int\int_{(\Omega)} \left\{ \left(\frac{\partial u_i}{\partial x}\right)^2 + \left(\frac{\partial u_i}{\partial y}\right)^2 \right\} dx\, dy$$

$$+ \sum_{i,j=1}^{n} \int\int_{(\Omega)} u_i \left\{ \frac{\partial (a_{ji} u_j)}{\partial x} + \frac{\partial (b_{ji} u_j)}{\partial y} - c_{ji} u_j \right\} dx\, dy = 0.$$

But

$$\sum_{i,j=1}^{n} \int\int_{(\Omega)} \left\{ u_i u_j \left(\frac{\partial a_{ji}}{\partial x} + \frac{\partial b_{ji}}{\partial y} - c_{ji} \right) + u_i \left(a_{ji} \frac{\partial u_j}{\partial x} + b_{ji} \frac{\partial u_j}{\partial y} \right) \right\} dx\, dy$$

$$= \tfrac{1}{2} \sum_{i,j=1}^{n} \int\int_{(\Omega)} u_i u_j \left(\frac{\partial a_{ji}}{\partial x} + \frac{\partial b_{ji}}{\partial y} - 2c_{ji} \right) dx\, dy$$

for, since the functions u_i vanish on the boundary,

$$\int\int_{(\Omega)} a_{ji} u_i \frac{\partial u_j}{\partial x}\, dx\, dy = -\int\int_{(\Omega)} \left(u_i u_j \frac{\partial a_{ji}}{\partial x} + u_j a_{ji} \frac{\partial u_i}{\partial x} \right) dx\, dy,$$

and similarly for the other terms. The above equation then becomes:

$$\sum_{i=1}^{n} \int\int_{(\Omega)} \left\{ \left(\frac{\partial u_i}{\partial x}\right)^2 + \left(\frac{\partial u_i}{\partial y}\right)^2 \right\} dx\, dy$$

$$+ \tfrac{1}{2} \sum_{i,j=1}^{n} \int\int_{(\Omega)} u_i u_j \left(\frac{\partial a_{ji}}{\partial x} + \frac{\partial b_{ji}}{\partial y} - 2c_{ji} \right) dx\, dy = 0.$$

The solutions of the homogeneous system (4) which vanish on the boundary are therefore all zero, if within Ω

$$\sum_{i,j=1}^{n} \left\{ u_i u_j \left(\frac{\partial a_{ji}}{\partial x} + \frac{\partial b_{ji}}{\partial y} - 2c_{ji} \right) \right\} > 0.$$

We have therefore the theorem:

The solutions of the system

$$L_i(u_1, \cdots, u_n) = f_i \qquad (i = 1, 2, \cdots, n)$$

within a region Ω are uniquely determined by their values on the boundary of Ω provided that the coefficients a_{ji}, b_{ji}, c_{ji} are such that

$$\sum_{i,j=1}^{n} i,j \left\{ u_i u_j \left(\frac{\partial a_{ji}}{\partial x} + \frac{\partial b_{ji}}{\partial y} - 2c_{ji} \right) \right\}$$

is a positive definite form.

§ 3. *Green's Functions and the Law of Reciprocity.*

By Green's functions of the system of differential equations

$$L_i(u_1, \cdots, u_n) = 0 \qquad (i = 1, 2, \cdots, n) \tag{4}$$

for a region Ω of the x, y plane, we shall understand n sets of n functions of x, y, ξ, η, $G_{ki}(x, y, \xi, \eta)$, which have the following properties:

Each set G_{ki} $(i = 1, 2, \cdots, n)$ *is, in the variables* x, y, *a solution of* (4) *within the region formed by excluding from* Ω *the point* $x = \xi, y = \eta$.

The functions have the form

$$G_{ki} = R_{ki}(x, y, \xi, \eta) \qquad (k \neq i),$$

$$G_{ii} = \log \sqrt{(x-\xi)^2 + (y-\eta)^2} + R_{ii}(x, y, \xi, \eta),$$

where the functions R *possess second derivatives with respect to* x *and* y *for all points within* Ω *and on the boundary.*

Each function vanishes identically in ξ, η *for values of* x, y *on the boundary.*

If the coefficients of the system (4) satisfy the condition of the theorem stated at the close of the preceding paragraph, the Green's functions are uniquely determined. For if two sets $G_{ki}^{(1)}$, $G_{ki}^{(2)}$ existed, the functions

$$u_i = G_{ki}^{(1)} - G_{ki}^{(2)}$$

for any k, would be solutions of (4) within the entire region Ω which vanish on the boundary.

Denote a set of Green's functions of the adjoint system

$$M_i(v_1, \cdots, v_n) = 0 \qquad (i = 1, 2, \cdots, n) \tag{5}$$

by $H_{ki}(x, y, \xi, \eta)$.

The Green's functions G_{ki} *of the system* (4) *and* H_{ki} *of the adjoint system* (5) *obey the following law of reciprocity:*

$$G_{ki}(x, y, \xi, \eta) = H_{ik}(\xi, \eta, x, y).$$

To prove this relation consider any two points ξ, η and ξ', η' within Ω, and substitute in (2) for $u_1, u_2, \cdots, u_n$ the functions $G_{ki}(x, y, \xi, \eta)$ $(i = 1, 2, \cdots, n)$ and for $v_1, v_2, \cdots, v_n$ the functions $H_{ik}(x, y, \xi', \eta')$ $(k = 1, 2, \cdots, n)$. Apply (2) to the region Ω' bounded by the boundary of Ω and two circles c, c' within Ω, of radii r, r' about ξ, η and ξ', η'. Since the functions G form a solution

of (4) and the functions H a solution of (5) within Ω', and all the functions vanish on the boundary, (2) becomes:

$$\sum_{j=1}^{n}\int_{(c+c')}\left\{H_{ij}(x,y,\xi',\eta')\frac{\partial G_{ki}(x,y,\xi,\eta)}{\partial n} - G_{ki}(x,y,\xi,\eta)\frac{\partial H_{ij}(x,y,\xi',\eta')}{\partial n}\right\}ds$$

$$+\sum_{j,k=1}^{n}\int_{(c+c')}G_{kh}(x,y,\xi,\eta)H_{ij}(x,y,\xi',\eta')\{a_{jh}\cos(n,x) + b_{jh}\cos(n,y)\}ds = 0.$$

This equation holds for all r, r' however small. Replacing n, ds by $-r, rd\theta$ on c and by $-r', r'd\theta'$ on c' we have:

$$\int_0^{2\pi}H_{ik}(x,y,\xi',\eta')\frac{\partial G_{kk}(x,y,\xi,\eta)}{\partial r}rd\theta - \int_0^{2\pi}G_{ki}(x,y,\xi,\eta)\frac{\partial H_{ii}(x,y,\xi',\eta')}{\partial r'}r'd\theta' + R + R' = 0,$$

where

$$\lim_{r=0}R = 0, \qquad \lim_{r'=0}R' = 0,$$

for the products of r or r' with all terms except $\partial G_{kk}/\partial r$ and $\partial H_{ii}/\partial r'$ approach zero as r or r' approaches zero, since these are the only terms that contain more than a logarithmic singularity. Passing to the limit $r = 0, r' = 0$ we obtain by well known methods*

$$H_{ik}(\xi,\eta,\xi',\eta') = G_{ki}(\xi',\eta',\xi,\eta),$$

which is the law of reciprocity.

From this law the theorem is evident:

If a set of Green's functions exists for the system (4) *and for the adjoint system* (5), *then these sets are uniquely determined.*

If the system (4) is *self adjoint* the law of reciprocity becomes

$$G_{ki}(x,y,\xi,\eta) = G_{ik}(\xi,\eta,x,y),$$

and it follows that the set is uniquely determined.

§ 3. *Integral Representation of Solutions.*

Let the u_i in (2) be a set of solutions of the equations

$$L_i(u_1,\cdots,u_n) = f_i \qquad (i=1,i,\cdots,n), \tag{3}$$

*See e. g. PICARD, *Traité d'analyse*, vol. 2, p. 15.

within Ω, and denote the values of u_i on the boundary S of Ω by $\bar{u}_i$. For v_i in (2) take the Green's functions $H_{ki}(x, y, \xi, \eta)$ of the adjoint system (5). Applying (2) to the region Ω' bounded by S and a circle c, within Ω, of radius r about ξ, η we have, after replacing n, ds on c by $-r$, $rd\theta$,

$$\sum_{i=1}^{n}\int\int_{(\Omega')} H_{ki}f_i\,dx\,dy = -\sum_{i=1}^{n}\int_{(S)} \bar{u}_i\frac{\partial H_{ki}}{\partial n}\,ds$$

$$-\sum_{i=1}^{n}\int_0^{2\pi}\left\{H_{ki}\frac{\partial u_i}{\partial r} - u_i\frac{\partial H_{ki}}{\partial r} + R\right\}rd\theta,$$

where

$$\lim_{r=0}(Rr) = 0.$$

In the limit $r = 0$ the term containing $\partial H_{kk}/\partial r$ is the only one which gives a value different from zero to the last integral. In the same manner as for a single equation we have then in the limit:

$$2\pi u_k(\xi, \eta) = \sum_{i=1}^{n}\int_{(S)} \bar{u}_i\frac{\partial H_{ki}}{\partial n}\,ds + \sum_{i=1}^{n}\int\int_{(\Omega)} H_{ki}f_i\,dx\,dy. \tag{6}$$

The integral formula (6) *gives the set of solutions of* (3) *within* Ω *which assume the values* $\bar{u}_i$ *on the boundary.*

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STUDIES IN THE GENERAL THEORY OF RULED SURFACES*

BY

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The congruence Γ, which is made of all the generators of the first kind on the osculating hyperboloids of a ruled surface, has a great many interesting properties. Some of them have been considered in a previous paper.‡ We shall continue the consideration of this congruence and of configurations associated with it, completing in this way some of our previous investigations very essentially. We shall also study the osculating linear complex, and the point-to-plane correspondence to which it gives rise, enabling us to generalize some well-known theorems of CREMONA and LIE.

For the most part we shall, in this paper, confine our attention to the general case, when the flecnode curve intersects every generator in two distinct points. The case of coincidence will be left for a future occasion, as it requires the use of a different normal form for the equations than that here adopted.

The notations are the same as in previous papers. To save space they are not again explained.

§ 1. *The derivative cubic curve.*

If α_1 and α_2 are arbitrary, $\alpha_1 y_k + \alpha_2 z_k$ will represent the coördinates of an arbitrary point on the generator g of the ruled surface, where (y_k, z_k) for $k = 1, 2, 3, 4$ are four simultaneous systems of solutions of our system of differential equations, whose determinant does not vanish. We shall usually write $\alpha_1 y + \alpha_2 z$, suppressing the index k, as in previous papers. Of course this is essentially a form of vector analysis, which enables us to make one equation do the work of four. The point $\alpha_1 \rho + \alpha_2 \sigma$ of the corresponding generator g' of S', will then be such that the line joining it to $\alpha_1 y + \alpha_2 z$ is a generator of the sec-

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‡ *On a certain congruence associated with a given ruled surface.* Transactions of the American Mathematical Society, vol. 4 (1903), pp. 185–200, hereafter referred to as *Congruence*. My other papers will be quoted by the initial words of their title. See *Covariants*, vol. 3 (1902), p. 423, footnote.

ond kind on the hyperboloid H osculating S along g. Therefore if β_1 and β_2 are arbitrary,

$$\beta_1(\alpha_1 y + \alpha_2 z) + \beta_2(\alpha_1 \rho + \alpha_2 \sigma)$$

will be an arbitrary point of H.

If we choose the tetrahedron $P_y P_z P_\rho P_\sigma$ as tetrahedron of reference for a system of homogeneous coördinates, we may take them so that the coördinates of any point represented by an expression of the form $\lambda y + \mu z + \nu\rho + \kappa\sigma$ will be $x_1 = \lambda$, $x_2 = \mu$, $x_3 = \nu$, $x_4 = \kappa$. We have then

$$x_1 = \beta_1\alpha_1, \qquad x_2 = \beta_1\alpha_2, \qquad x_3 = \beta_2\alpha_1, \qquad x_4 = \beta_2\alpha_2$$

as the coördinates of an arbitrary point on H, and therefore

$$x_1 x_4 - x_2 x_3 = 0 \tag{1}$$

as the equation of H in this system of coördinates.

Let us consider now the hyperboloid H', which osculates S' along g'. The coördinates of P_ρ and P_σ were obtained from the system of differential equations defining S by forming

$$\rho = 2y' + p_{11}y + p_{12}z, \qquad \sigma = 2z' + p_{21}y + p_{22}z.$$

We shall obtain the coördinates of two points on a generator g'' of the derivative of S' with respect to x, by applying the same process to the equations [*Congruence*, (15)] which define S'. The ruled surface S'' thus obtained shall be naturally called the second derivative of S with respect to x. Its generator g'' is then a generator of the hyperboloid H' which osculates S' along g'. The following quantities

$$\begin{aligned} 2\rho' + P_{11}\rho + P_{12}\sigma &= u_{11}y + u_{12}z + \lambda_{11}\rho + \lambda_{12}\sigma, \\ 2\sigma' + P_{21}\rho + P_{22}\sigma &= u_{21}y + u_{22}z + \lambda_{21}\rho + \lambda_{22}\sigma \end{aligned} \tag{2}$$

are the coördinates of two points on g''.

These equations show that g'' intersects g, if and only if $\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21} = 0$, i. e., if $K = v_{11}v_{22} - v_{12}v_{21} = 0$, provided we assume that S' is not developable. By changing the independent variable one can always change K into $\overline{K}$ such that $\overline{K} = 0$. The equation for η, [*Congruence*, equ. (4)], which must be satisfied so as to make $\overline{K} = 0$, is of the second order. *Therefore there exist ∞^2 non-developable ruled surfaces in the congruence Γ, each of which, when considered as the first derivative of S, gives rise to a second derivative whose generators intersect the corresponding generators of S.*

Let us consider any point on g', whose coördinates are $\epsilon_1\rho + \epsilon_2\sigma$. The corresponding point on g'' will be given by

$$(\epsilon_1 u_{11} + \epsilon_2 u_{21})y + (\epsilon_1 u_{12} + \epsilon_2 u_{22})z + (\epsilon_1\lambda_{11} + \epsilon_2\lambda_{21})\rho + (\epsilon_1\lambda_{12} + \epsilon_2\lambda_{22})\sigma.$$

Therefore, the expression

$$\delta_1(\epsilon_1 u_{11} + \epsilon_2 u_{21})y + \delta_1(\epsilon_1 u_{12} + \epsilon_2 u_{22})z + [\delta_1(\epsilon_1\lambda_{11} + \epsilon_2\lambda_{21}) + \delta_2\epsilon_1]\rho + [\delta_1(\epsilon_1\lambda_{12} + \epsilon_2\lambda_{22}) + \delta_2\epsilon_2]\sigma$$

will, for arbitrary values of ϵ_1, ϵ_2, δ_1, δ_2, represent an arbitrary point of H'. If we introduce again our special system of coördinates we have

$$x_1 = u_{11}\delta_1\epsilon_1 + u_{21}\delta_1\epsilon_2, \qquad x_3 = \lambda_{11}\delta_1\epsilon_1 + \lambda_{21}\delta_1\epsilon_2 + \delta_2\epsilon_1,$$
$$x_2 = u_{12}\delta_1\epsilon_1 + u_{22}\delta_1\epsilon_2, \qquad x_4 = \lambda_{21}\delta_1\epsilon_1 + \lambda_{22}\delta_1\epsilon_2 + \delta_2\epsilon_2,$$

as the coördinates of an arbitrary point of H'. If we eliminate δ_1, δ_2, ϵ_1, ϵ_2, we find *the equation of H':*

$$(3)\quad \begin{aligned}&(u_{22}x_1 - u_{21}x_2)[Jx_4 - (\lambda_{12}u_{22} - \lambda_{22}u_{12})x_1 + (\lambda_{12}u_{21} - \lambda_{22}u_{11})x_2] \\ &+ (u_{12}x_1 - u_{11}x_2)[Jx_3 - (\lambda_{11}u_{22} - \lambda_{21}u_{12})x_1 + (\lambda_{11}u_{21} - \lambda_{21}u_{11})x_2] = 0.\end{aligned}$$

We shall mostly assume that P_y and P_z are the flecnodes of g, so that $u_{12} = u_{21} = 0$, and (3) simplifies into

$$(3a)\quad -\lambda_{12}u_{22}^2x_1^2 + \lambda_{21}u_{11}^2x_2^2 + u_{11}u_{22}(\lambda_{11} - \lambda_{22})x_1x_2 + u_{11}u_{22}(u_{22}x_1x_4 - u_{11}x_2x_3) = 0.$$

It is easy to see from (3a) that H' cannot coincide with H, unless S is a quadric.

The hyperboloids H and H' have the straight line g' in common. The rest of their intersection is therefore, in general, a space cubic. We shall call it the *derivative cubic*, and discuss some of its properties. It is interesting to notice that *we obtain in this way associated with every ruled surface, a surface containing a single infinity of twisted cubics.* This surface we shall also consider.

Let us again assume that P_y and P_z are the flecnodes of g. Then, it follows from (1) and (3a) that we may take

$$(4)\quad \begin{aligned}&x_1 = tx_2, \qquad x_2 = J(u_{11} - u_{22})t, \qquad x_3 = tx_4, \\ &x_4 = -\lambda_{12}u_{22}^2t^2 + u_{11}u_{22}(\lambda_{11} - \lambda_{22})t + \lambda_{21}u_{11}^2\end{aligned}$$

as the parametric equations of the cubic. From these, the following corollaries follow at once. *If $\lambda_{11} - \lambda_{22} = 0$, the derivative cubic, and therefore the hyperboloid H', intersects g in two points, which are harmonic conjugates with respect to the flecnodes. If*

$$(\lambda_{11} - \lambda_{21})^2 + 4\lambda_{12}\lambda_{21} = 0,$$

the cubic is tangent to g. The congruence Γ contains ∞^2 surfaces S' corresponding to each of these properties of the derivative cubic. For, the corresponding equations for η are again of the second order.

The equation of a plane which is tangent to H' at a point (x_1', x_2', x_3', x_4'), P_y and P_z being flecnodes, is

$$
\begin{aligned}
&[-2\lambda_{12}u_{22}^2x_1' + u_{11}u_{22}(\lambda_{11}-\lambda_{22})x_2' + u_{11}u_{22}^2x_4']x_1 \\
(5)\qquad &+ [u_{11}u_{22}(\lambda_{11}-\lambda_{22})x_1' + 2\lambda_{21}u_{11}^2x_2' - u_{11}^2u_{22}x_3']x_2 \\
&- u_{11}^2u_{22}x_2'x_3 + u_{11}u_{22}^2x_1'x_4 = 0.
\end{aligned}
$$

Consider the two points of g'' which correspond to P_y and P_z, viz.,

$$P_1'' = (u_{11}, 0, \lambda_{11}, \lambda_{12}), \qquad P_2'' = (0, u_{22}, \lambda_{21}, \lambda_{22}).$$

The equations of the two planes tangent to S'' or H' at these points respectively, are

$$-u_{22}\lambda_{12}x_1 - u_{11}\lambda_{22}x_2 + u_{11}u_{22}x_4 = 0, \qquad u_{22}\lambda_{11}x_1 + u_{11}\lambda_{21}x_2 - u_{11}u_{22}x_3 = 0.$$

They intersect g, i. e., the line $x_3 = x_4 = 0$, in two points

$$P_1 = (u_{11}\lambda_{22}, -u_{22}\lambda_{12}, 0, 0), \qquad P_2 = (u_{11}\lambda_{21}, -u_{22}\lambda_{11}, 0, 0),$$

which coincide, if and only if $\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21} = 0$, i. e., if and only if g'' intersects g, as is moreover geometrically evident. P_1 and P_2 are harmonic conjugates with respect to the flecnodes if $\lambda_{11}\lambda_{22} + \lambda_{12}\lambda_{21} = 0$, i. e., *under this condition the planes, tangent to the second derivative at the points which correspond to the flecnodes of g, divide these flecnodes harmonically.*

The line P_zP_ρ has, besides P_ρ, another point in common with H'. Its coördinates are found to be $(0, u_{22}, \lambda_{21}, 0)$. Similarly P_yP_σ intersects H' in a second point $(u_{11}, 0, 0, \lambda_{12})$. Join these two points. The coördinates of any point on this line joining them will be $(\mu u_{11}, \lambda u_{22}, \lambda\lambda_{21}, \mu\lambda_{12})$, where $\lambda:\mu$ determines the position of the particular point. It is easily seen, by substituting in (3a), that this line is entirely on H' if $\lambda_{11} - \lambda_{22} = 0$, and in no other case, provided that S' is not developable. We have therefore the following result. *Corresponding to every point of S we have a point of S'. If each of the two flecnodes on a generator g of S be joined to the point of S' which corresponds to the other, the two straight lines thus obtained intersect the hyperboloid H' osculating S' along g' in two new points. The line joining these latter points lies entirely on H', if H' intersects g in two points which are harmonic conjugates with respect to the flecnodes. The converse of the theorem is also true.*

Let us introduce the following abbreviations:

$$
\begin{aligned}
&A = 2u_{11}u_{22}(u_{11} - u_{22}), \qquad \psi = Bt_1^2 + 2Ct_1t_2 + Dt_2^2, \\
(6)\qquad &B = v_{12}u_{22}, \qquad C = \tfrac{1}{2}(u_{11}v_{22} - u_{22}v_{11}), \qquad D = -v_{21}u_{11},
\end{aligned}
$$

and let us write the parameter t of the cubic curve in homogeneous form. Then we may write instead of (4),

$$
\begin{aligned}
&x_1 = At_1^2t_2, \qquad x_3 = (Bt_1^2 + 2Ct_1t_2 + Dt_2^2)t_1 = \psi t_1, \\
(7)\qquad &x_2 = At_1t_2^2, \qquad x_4 = (Bt_1^2 + 2Ct_1t_2 + Dt_2^2)t_2 = \psi t_2.
\end{aligned}
$$

If the cubic degenerates, each irreducible part will be a plane curve (a conic or a straight line). If therefore the cubic degenerates it must be possible to satisfy the equation

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0,$$

for all values of $t_1 : t_2$, the coefficients being independent of $t_1 : t_2$. If we substitute in this equation the values (7), and equate to zero the coefficients of t_1^3, $t_1^2 t_2$, etc., we find

$$(8)\quad a_3B = 0,\ a_4D = 0,\ a_1A + 2a_3C + a_4B = 0,\ a_2A + a_3D + 2a_4C = 0.$$

Let us assume first that neither B nor D vanishes. Then $a_3 = a_4 = 0$, and $A = 0$, for if A were not zero, we would have also $a_1 = a_2 = 0$, i. e., there would be no plane containing the (supposedly) degenerate cubic. But from $A = 0$ follows either $u_{11} - u_{22} = 0$, which would make S a quadric, or else u_{11} or u_{22} would vanish, which however contradicts the assumption that B and D shall not be zero.

Let us now assume $B \neq 0$, $D = 0$. Then $a_3 = 0$, and either u_{11} or v_{21} must vanish, i. e., either S' is developable or S has a straight line directrix. Similarly if $B = 0$, $D \neq 0$. Finally if $B = 0$, $D = 0$, either S has two straight line directrices, or else it has one while S' is developable. We have therefore the following theorem. *If the surface has one or more straight line directrices the derivative cubic always degenerates. In all other cases, the only way to obtain a degenerate derivative cubic consists in taking as derivative ruled surface of S, one of the developables of the congruence* Γ.

Another question at once suggests itself. To every value of x, i. e., to every generator of S there belongs a derivative cubic. In general, the cubics belonging to values of x, differing from each other by an infinitesimal δx, will not intersect. Their shortest distance will be an infinitesimal of the same order as δx. It may happen however that, for an appropriately chosen variable, this distance becomes infinitesimal of a higher order, or as we may say briefly, that consecutive cubics intersect. We ask now: is it possible to choose the independent variable in such a way that every pair of consecutive derivative cubics may intersect?

By putting $y = y_k$, $z = z_k$ $(k = 1, 2, 3, 4)$ in

$$(9)\qquad \phi = At_1t_2(t_1y + t_2z) + \psi(t_1\rho + t_2\sigma),$$

we obtain the coördinates of any point P_ϕ on the cubic. As x changes we go from one cubic to another; as $t_1 : t_2$ changes we go from one point on a certain cubic to another point of the same curve. Equation (9) gives therefore, if both x and $t_1 : t_2$ be taken as variables, the locus of all such points P_ϕ, i. e., the surface generated by all of the derivative cubics of S. If $t_1 : t_2$ be chosen as a function of x, a curve is picked out upon this surface. Let us differentiate ϕ

totally, i. e., assuming that t_1 and t_2 are functions of x, and consider the quantity

$$\phi + \frac{d\phi}{dx}\delta x,$$

where δx is an infinitesimal. This will clearly represent the coördinates of a point on the adjacent derivative cubic determined by the parameters $x + \delta x$ and $t_k + dt_k/dx\,\delta x$. If the original cubic and this second one, infinitesimally close to it, intersect, it must be possible to choose t_k as functions of x in such a way, that the corresponding points of the two curves coincide up to infinitesimals of higher than the first order. Therefore $d\phi/dx$ must differ from a multiple of ϕ only by an infinitesimal quantity. Proceeding to the limit we must therefore have

$$\frac{d\phi}{dx} = \omega\phi, \tag{10}$$

We find by differentiation

$$\begin{aligned}\frac{d\phi}{dx} = {} & [(t_1t_2' + t_1't_2)A + A't_1t_2](t_1y + t_2z) \\ & + y[At_1t_2(t_1' - \tfrac{1}{2}p_{11}t_1 - \tfrac{1}{2}p_{21}t_2) + \tfrac{1}{2}\psi u_{11}t_1] \\ & + z[At_1t_2(t_2' - \tfrac{1}{2}p_{12}t_1 - \tfrac{1}{2}p_{22}t_2) + \tfrac{1}{2}\psi u_{22}t_2] \\ & + \rho[\tfrac{1}{2}At_1^2t_2 + \psi(t_1' - \tfrac{1}{2}p_{11}t_1 - \tfrac{1}{2}p_{21}t_2) + \psi't_1] \\ & + \sigma[\tfrac{1}{2}At_1t_2^2 + \psi(t_2' - \tfrac{1}{2}p_{12}t_1 - \tfrac{1}{2}p_{22}t_2) + \psi't_2],\end{aligned} \tag{10a}$$

denoting as usual differentiation by strokes.

If we substitute in (10) we find the following four equations:

$$\begin{aligned}&(a)\ [(t_1t_2' + t_1't_2)A + A't_1t_2]t_1 + At_1t_2(t_1' - \tfrac{1}{2}p_{11}t_1 - \tfrac{1}{2}p_{21}t_2) \\ &\qquad + \tfrac{1}{2}\psi u_{11}t_1 = \omega At_1^2t_2, \\ &(b)\ [(t_1t_2' + t_1't_2)A + A't_1t_2]t_2 + At_1t_2(t_2' - \tfrac{1}{2}p_{12}t_1 - \tfrac{1}{2}p_{22}t_2) \\ &\qquad + \tfrac{1}{2}\psi u_{22}t_2 = \omega At_1t_2^2, \\ &(c)\ \tfrac{1}{2}At_1^2t_2 + \psi(t_1' - \tfrac{1}{2}p_{11}t_1 - \tfrac{1}{2}p_{21}t_2) + \psi't_1 = \omega\psi t_1, \\ &(d)\ \tfrac{1}{2}At_1t_2^2 + \psi(t_2' - \tfrac{1}{2}p_{12}t_1 - \tfrac{1}{2}p_{22}t_2) + \psi't_2 = \omega\psi t_2.\end{aligned} \tag{11}$$

If we multiply both members of (a) by t_2, of (b) by $-t_1$, and add, and if we treat (c) and (d) in the same way, we find

$$\begin{aligned}&(e)\ At_1t_2^2(t_1' - \tfrac{1}{2}p_{11}t_1 - \tfrac{1}{2}p_{21}t_2) - At_1^2t_2(t_2' - \tfrac{1}{2}p_{12}t_1 - \tfrac{1}{2}p_{22}t_2) \\ &\qquad + \tfrac{1}{2}\psi t_1t_2(u_{11} - u_{22}) = 0, \\ &(f)\ \psi t_2(t_1' - \tfrac{1}{2}p_{11}t_1 - \tfrac{1}{2}p_{21}t_2) - \psi t_1(t_2' - \tfrac{1}{2}p_{12}t_1 - \tfrac{1}{2}p_{22}t_2) = 0.\end{aligned}$$

Let us assume first $\psi \neq 0$. Divide both members of (f) by ψ, and compare with (e). We find

$$\psi t_1 t_2 (u_{11} - u_{22}) = 0,$$

i. e., either S is a quadric, or else either t_1 or t_2 must vanish identically. Assume $t_1 = 0$. Then (e) is satisfied. From (f) we would find $p_{21} = 0$, for t_2 cannot be zero, since the ratio $t_1 : t_2$ is the parameter which determines a point on the cubic. But from $u_{21} = p_{21} = 0$ follows further $v_{21} = 0$, i. e., $D = 0$, so that in this case $\psi = Bt_1^2 + 2Ct_1 t_2 = 0$ contrary to our assumption. It is therefore impossible to satisfy (11) except by putting $\psi = 0$. According to (c) and (d) this gives either $t_1 = 0$, or $t_2 = 0$, or $A = 0$. Assume $A \neq 0$ and say $t_1 = 0$. All equations (11) are now satisfied. But from $t_1 = 0$, $\psi = 0$ follows $D = 0$, i. e., either u_{11} or v_{21} must vanish, i. e., either S has a straight line directrix or else S' is developable. If however $A = 0$, either S is a quadric or else S' is developable.

In connection with our last theorem, we may therefore say: *It is impossible to choose the surface S' of the congruence Γ in such a way that consecutive derivative cubics may intersect, except in the trivial cases when the cubics are degenerate.*

If in (9) we put $\psi = 0$, we obtain the locus of the intersections of the cubic with the generator of S to which it belongs, i. e., a certain curve cutting every generator twice. This may be an asymptotic curve. It is, in fact, if the further conditions

$$(12) \qquad 2t_1' - p_{11}t_1 - p_{21}t_2 = \omega t_1, \qquad 2t_2' - p_{12}t_1 - p_{22}t_2 = \omega t_2$$

are satisfied, where ω is arbitrary. For, as $(10a)$ shows, the line joining P_ϕ to $P_{\phi+\phi'\delta x}$, i. e., the tangent to this curve, is then a generator of the second kind on the hyperboloid osculating S along g. In other words it is a tangent to an asymptotic curve of S.

In general, of course the conditions (12) and $\psi = 0$ can not both be satisfied at once. The question is: when are these conditions consistent? We find from (12)

$$t_1' = \tfrac{1}{2}[(p_{11} + \omega)t_1 + p_{21}t_2], \qquad t_2' = \tfrac{1}{2}[p_{12}t_1 + (p_{22} + \omega)t_2].$$

Let us substitute these values of t_1' and t_2' in the equation $d\psi/dx = 0$. We shall find

$$[B' + B(p_{11} + \omega) + Cp_{12}]t_1^2 + [D' + D(p_{22} + \omega) + Cp_{21}]t_2^2 + [2C' + Bp_{12} + Dp_{21} + C(p_{11} + p_{22} + 2\omega)]t_1 t_2 = 0,$$

which compared with $\psi = 0$ gives,

$$B' + B(p_{11} + \omega) + Cp_{12} = \tau B,$$

$$D' + D(p_{22} + \omega) + Cp_{21} = \tau D,$$

$$2C' + Bp_{12} + Dp_{21} + C(p_{11} + p_{22} + 2\omega) = 2\tau C,$$

where τ is a proportionality factor. Eliminating τ, we find

$$(13)\quad \begin{aligned} &(a)\ 2(B'C - BC') + BC(p_{11} - p_{22}) + (2C^2 - BD)p_{12} - B^2 p_{21} = 0,\\ &(b)\ 2(D'C - DC') + DC(p_{22} - p_{11}) + (2C^2 - BD)p_{21} - D^2 p_{12} = 0,\\ &(c)\ BD' - B'D + BD(p_{11} - p_{22}) + C(Dp_{12} - Bp_{21}) = 0. \end{aligned}$$

We may always assume that our system of differential equations has been so written that $p_{11} - p_{22} = 0$. Equations (13) may then be regarded as three homogeneous linear equations for $1, p_{12}, p_{21}$. Their determinant must therefore be zero. This gives rise to three alternatives: either $C^2 - BD$ or C or $BD' - B'D$ must vanish.

Consider first the case $C^2 - BD = 0$. Equations (13) become

$$(14)\quad \begin{gathered} 2(B'C - BC') - B^2 p_{21} = 0, \qquad 2(D'C - DC') - D^2 p_{12} = 0,\\ BD' - B'D + CDp_{12} - CBp_{21} = 0. \end{gathered}$$

If we multiply both members of the first two equations by CD and $-CB$ respectively, add and make use of $C^2 = BD$, we find

$$2C^2(DB' - D'B) + C^2(CDp_{12} - CBp_{21}) = 0,$$

whence, if $C \neq 0$,

$$-2(BD' - B'D) + CDp_{12} - CBp_{21} = 0.$$

But if we consider the last equation of (14), this gives $BD' - B'D = 0$, $Dp_{12} - Bp_{21} = 0$, which last equation may be written

$$p_{12}p_{21}(u_{11} - u_{22})^2 = 0,$$

i. e., the surface S has at least one straight line directrix.

In the second place let $C = 0$. Then (13) becomes

$$B(Dp_{12} + Bp_{21}) = 0, \qquad D(Dp_{12} + Bp_{21}) = 0, \qquad BD' - B'D = 0.$$

According to the first two equations S must either have at least two straight line directrices, or else $u_{11} + u_{22}$ must be zero. The third condition gives, on substituting the values of B and D,

$$\frac{u'_{22}}{u_{22}} - \frac{u'_{11}}{u_{11}} + \frac{v'_{12}}{v_{12}} - \frac{v'_{21}}{v_{21}} = 0,$$

if we assume that S has no straight line directrix, so that v_{12} and v_{21} are not

zero. By integration we find

$$\frac{v_{12}}{v_{21}}\frac{u_{22}}{u_{11}} = \text{const.},$$

whence, since $u_{11} + u_{22} = 0$ in this case,

$$\frac{p_{12}}{p_{21}} = \text{const.}$$

But if we substitute into the condition $\Delta = 0$ for a surface belonging to a linear complex the assumptions $u_{12} = u_{21} = p_{11} - p_{22} = 0$, which we have made, we find either $u_{11} - u_{22} = 0$, i. e., S is a quadric, or $p_{12}p'_{21} - p'_{12}p_{21} = 0$, i. e., $p_{12}/p_{21} = \text{const.}$ Therefore we see that if $C = 0$, S either has straight line directrices, or at least belongs to a linear complex. Moreover in that case the independent variable must be so chosen as to make $u_{11} + u_{22}$ vanish.

If finally $BD' - B'D = 0$, we find from (13) either $C = 0$, which leads us back to the case just considered, or else $Dp_{12} - Bp_{21}$ must vanish, which gives again a surface S with at least one straight line directrix.

In all of these cases the surface S belongs to a linear complex. In the special cases the asymptotic lines thus obtained are merely two of the straight line directrices, if there are two, or else one straight line directrix and another curved asymptotic line obtained by putting $u_{11} + u_{22} = 0$. If we leave aside the trivial cases, we can say that *if a ruled surface belongs to a linear complex, and if the independent variable is so chosen that* $u_{11} + u_{22} = 0$, *the surface of derivative cubics determines an asymptotic curve upon it which intersects every generator twice.*

We have seen (*Congruence*, p. 197), that there exists in the congruence Γ a single infinity of ruled surfaces S' for which $u_{11} + u_{22} = 0$. They are those ruled surfaces of Γ whose intersections with the flecnode surface of S are asymptotic lines upon them. *But we can also characterize them by saying that such a surface is made up of the lines of* Γ *intersecting any asymptotic curve on the flecnode surface of* S. In fact, if we assume $u_{12} = u_{21} = 0$, $p_{11} = p_{22} = 0$, we may write the equations of the sheet F' of the flecnode surface as follows:*

$$(15)\qquad \begin{aligned} &y'' - 2\frac{q_{12}}{p_{12}}y' - \rho' - q_{11}y + \frac{q_{12}}{p_{12}}\rho = 0, \\ &\rho'' + [2(q_{11}+q_{22}) - p_{12}p_{21}]y' - 2\frac{q_{12}}{p_{12}}\rho' + [2q'_{11} - p_{12}q_{21} - 4\frac{q_{12}}{p_{12}}q_{11}]y - q_{22}\rho = 0. \end{aligned}$$

If $u_{11} + u_{22} = 0$, the coefficient of y' in the second equation vanishes, which proves that the curve C_ρ is an asymptotic curve on F'. *The asymptotic curves*

* The system of equations for the flecnode surface, in *Congruence*, p. 190, equ. (12), does not agree exactly with (15) because there the further assumption $u_{11} = 0$ was made. This should have been stated there.

*on F' and F'' must therefore correspond to each other. The congruence Γ is therefore a so-called W-congruence.**

We may now state our theorem as follows: *In order that the surface of derivative cubics may intersect the ruled surface S in an asymptotic curve, S must belong to a linear complex. Moreover, its derivative ruled surface must intersect the flecnode surface of S along an asymptotic curve.*

The asymptotic curve on S, thus determined, is unique. For the ratio $B:D$ which determines it cannot be changed by any transformation of the variables subject to the conditions $p_{11}-p_{22}=u_{12}=u_{21}=u_{11}+u_{22}=0$. We may then take any asymptotic curve of the flecnode surface and consider the ruled surface of Γ made up of the lines which intersect it. We obtain thus as a consequence a single infinity of surfaces made up of derivative cubics. All of of these intersect S along the same asymptotic curve.

But we notice further that $C=0$. Therefore, *this asymptotic curve intersects every generator in two points which are harmonic conjugates with respect to the flecnodes.* We shall meet this special asymptotic curve again in a later paragraph.

Another question suggests itself. Is it possible to choose the independent variable of our system in such a way that the derivative cubics shall be asymptotic lines on the surface generated by them?

In order to answer this question, we must first find the coördinates of the osculating plane of the cubic at any one of its points. For the moment we prefer to take the equation of the cubic referred to a non-homogeneous parameter t, i. e., in the form

$$x_1=At^2, \qquad x_2=At, \qquad x_3=Bt^3+2Ct^2+Dt, \qquad x_4=Bt^2+2Ct+D.$$

Its intersections with the plane $\sum u_i x_i=0$, whose coördinates are $(u_1,\cdots u_4)$, will be given by solving the cubic equation

$$Bu_3t^3+(Au_1+2Cu_3+Bu_4)t^2+(Au_2+2Cu_4+Du_3)t+Du_4=0.$$

The three roots of this cubic must coincide if the plane is an osculating plane of the cubic curve. They must therefore also satisfy the equations obtained from the above by twofold differentiation with respect to t. This gives the following conditions:

$$\begin{aligned} 3Bu_3t+Au_1+2Cu_3+Bu_4&=0,\\ (Au_1+2Cu_3+Bu_4)t+Au_2+2Cu_4+Du_3&=0,\\ (Au_2+2Cu_4+Du_3)t+3Du_4&=0. \end{aligned}$$

Of course, only the ratios of $u_1\cdots u_4$ are of interest, so that we may multiply $u_1\cdots u_4$ by a common factor if we please. We find

* Cf. BIANCHI-LUKAT, *Vorlesungen über Differentialgeometrie*, p. 315.

$$(16)\qquad \begin{aligned} u_1 &= B^2t^3 - 3BDt - 2CD, & u_3 &= AD,\\ u_2 &= 2BCt^3 + 3BDt^2 - D^2, & u_4 &= -ABt^3, \end{aligned}$$

as the coördinates of the plane osculating the derivative cubic belonging to the generator g at the point whose parameter is t, or in homogeneous form,

$$(17)\qquad \begin{aligned} u_1 &= B^2t_1^3 - 3BDt_1t_2^2 - 2CDt_2^3, & u_3 &= ADt_2^3,\\ u_2 &= 2BCt_1^3 + 3BDt_1^2t_2 - D^2t_2^3, & u_4 &= -ABt_1^3. \end{aligned}$$

We have already made use of the expression

$$\phi = At_1t_2(t_1y + t_2z) + \psi(t_1\rho + t_2\sigma)$$

as giving the coördinates of a point $(t_1 : t_2)$ of the cubic curve which belongs to the argument x, or as giving the coördinates of a point $(x, t_1 : t_2)$ of the surface formed by the aggregate of all of these curves. The plane which is tangent to this surface at the point $(x, t_1 : t_2)$ must contain also the point $t_1 : t_2$ of the adjacent cubic, i. e., the point whose coördinates are given by

$$\phi + \frac{\partial\phi}{\partial x}\delta x,$$

where in forming $\partial\phi/\partial x$, x, t_1 and t_2 are regarded as independent variables. The tangent plane must therefore contain the point whose coördinates are $\partial\phi/\partial x$. We have

$$\begin{aligned} \frac{\partial\phi}{\partial x} &= y[A't_1^2t_2 - \tfrac{1}{2}At_1t_2(p_{11}t_1 + p_{21}t_2) + \tfrac{1}{2}\psi u_{11}t_1]\\ &\quad + z[A't_1t_2^2 - \tfrac{1}{2}At_1t_2(p_{12}t_1 + p_{22}t_2) + \tfrac{1}{2}\psi u_{22}t_2]\\ &\quad + \rho[\tfrac{1}{2}At_1^2t_2 - \tfrac{1}{2}\psi(p_{11}t_1 + p_{21}t_2) + \frac{\partial\psi}{\partial x}t_1]\\ &\quad + \sigma[\tfrac{1}{2}At_1t_2^2 - \tfrac{1}{2}\psi(p_{12}t_1 + p_{22}t_2) + \frac{\partial\psi}{\partial x}t_2]. \end{aligned}$$

The point, whose coördinates $(\xi_1, \cdots, \xi_4)$ are the coefficients of y, z, ρ, σ in this expression, must be in the tangent plane of the point $(x, t_1 : t_2)$. If therefore, the cubic curve is an asymptotic line upon the surface, its osculating plane must contain the point $(\xi_1, \cdots, \xi_4)$, i. e., we must have

$$(18)\qquad u_1\xi_1 + u_2\xi_2 + u_3\xi_3 + u_4\xi_4 = 0.$$

We find in the first place

$$(19)\qquad \begin{aligned} \xi_1 &= u_{11}Bt_1^3 + (2A' + 2u_{11}C - Ap_{11})t_1^2t_2 + (u_{11}D - Ap_{21})t_1t_2^2,\\ \xi_2 &= u_{22}Dt_2^3 + (u_{22}B - Ap_{12})t_1^2t_2 + (2A' + 2u_{22}C - Ap_{22})t_1t_2^2, \end{aligned}$$

$$
\begin{aligned}
\xi_3 &= (2B' - Bp_{11})t_1^3 + (A - Bp_{21} + 4C' - 2Cp_{11})t_1^2 t_2 \\
&\quad + (2D' - Dp_{11} - 2Cp_{21})t_1 t_2^2 - Dp_{21} t_2^3, \\
\xi_4 &= -Bp_{12} t_1^3 + (2B' - Bp_{22} - 2Cp_{12})t_1^2 t_2 \\
&\quad + (A - Dp_{12} + 4C' - 2Cp_{22})t_1 t_2^2 + (2D' - Dp_{22})t_2^3.
\end{aligned} \tag{19}
$$

If these values, and the values (17) for $u_1, \cdots, u_4$ be substituted in (18), and the coefficients of $t_1^6, t_1^5 t_2$, etc., be successively equated to zero, the following seven equations make their appearance:

$$
\begin{aligned}
&u_{11}B^3 + AB^2 p_{12} = 0, \\
&B^2(2A' + 2u_{11}C - Ap_{11}) + 2BC(u_{22}B - Ap_{12}) \\
&\qquad - AB(2B' - Bp_{22} - 2Cp_{12}) = 0, \\
&B^2(u_{11}D - Ap_{21}) - 3B^2 Du_{11} + 2BC(2A' + 2u_{22}C - Ap_{22}) \\
&\qquad + 3BD(u_{22}B - Ap_{12}) - AB(A - Dp_{12} + 4C' - 2Cp_{22}) = 0, \\
&-3BD(2A' + 2u_{11}C - Ap_{11}) - 2BCD(u_{11} - u_{22}) \\
&\qquad + 3BD(2A' + 2u_{22}C - Ap_{22}) \\
&\qquad + AD(2B' - Bp_{11}) - AB(2D' - Dp_{22}) = 0, \\
&-3BD(u_{11}D - Ap_{21}) - 2CD(2A' + 2u_{11}C - Ap_{11}) \\
&\quad + 3BD^2 u_{22} - D^2(u_{22}B - Ap_{12}) + AD(A - Bp_{21} + 4C' - 2Cp_{11}) = 0, \\
&-2CD(u_{11}D - Ap_{21}) - D^2(2A' + 2u_{22}C - Ap_{22}) \\
&\qquad + AD(2D' - Dp_{11} - 2Cp_{21}) = 0, \\
&-u_{22}D^3 - AD^2 p_{21} = 0.
\end{aligned} \tag{20}
$$

Let us assume that both B and D are different from zero. Then the first and last equations give

$$u_{11}u_{22}v_{12} + 2u_{11}u_{22}(u_{11} - u_{22})p_{12} = 0, \qquad u_{11}u_{22}v_{21} - 2u_{11}u_{22}(u_{11} - u_{22})p_{21} = 0,$$

or

$$u_{11}u_{22}(u_{11} - u_{22})p_{12} = u_{11}u_{22}(u_{11} - u_{22})p_{21} = 0.$$

But all of the possibilities here suggested give zero values to either B or D, or both, except $u_{11} - u_{22} = 0$, in which case S is a quadric. But this is only apparently an exception, arising from the fact that in this case the flecnode curve is indeterminant. If two of the straight lines of the second set be taken for the curves C_y and C_z, we have in this case also $B = D = 0$.

There remain two possibilities. If $B = D = 0$ all of the equations (20) are

satisfied. If S' is not developable, S must then belong to a linear congruence. If S' is developable, it is sufficient for S to have one straight line directrix. In either case the derivative cubic degenerates into a straight line, and is therefore obviously an asymptotic curve upon the surface of cubics.

Finally, it might happen that one only of the two quantities B and D is zero. Say $B = 0$, $D \neq 0$. Then the first four equations of (20) are satisfied. The other three become

$$-4u_{11}C^2 + ADp_{12} + A^2 = 0,$$

$$-2Cu_{11}D - D(2A' + 2u_{22}C - Ap_{22}) + A(2D' - Dp_{11}) = 0,$$

$$-u_{22}D - Ap_{21} = 0.$$

From the last of these we find

$$u_{11}u_{22}(u_{11} - u_{22})p_{21} = 0,$$

whence, since $D \neq 0$, follows $u_{22} = 0$, and therefore $A = 0$. The second equation now gives us $C = 0$, which also satisfies the first. We have again S' a developable surface, so that the cubic degenerates. Therefore

The derivative cubics of a ruled surface are asymptotic curves, upon the surface formed by their totality, only in the trivial cases when they degenerate into straight lines.

§ 2. *Null-system of the derivative cubic.*

A twisted cubic always determines a null-system, i. e., a point-to-plane correspondence with incident elements. Geometrically this correspondence may be set up as follows. An arbitrary plane intersects the curve in three points. The three planes, which osculate the curve in these points, intersect again in a point which is situated in the original plane. This is the point which corresponds to the plane in the null-system of the cubic.

We shall now set up the equations for this null-system. For this purpose it is more convenient to use the equations of the cubic referred to a non-homogeneous parameter t.

Let t_1, t_2, t_3 be the three values of t which correspond to the three points in which the plane

$$v_1x_1 + v_2x_2 + v_3x_3 + v_4x_4 = 0$$

intersects the cubic. Then t_1, t_2, t_3 are the roots of the cubic equation

$$Bv_3t^3 + (Av_1 + 2Cv_3 + Bv_4)t^2 + (Av_2 + Dv_3 + 2Cv_4)t + Dv_4 = 0.$$

Therefore we shall have

$$\frac{Av_1 + 2Cv_3 + Bv_4}{Bv_3} = -(t_1 + t_2 + t_3),$$

$$\frac{Av_2 + Dv_3 + 2Cv_4}{Bv_3} = t_2t_3 + t_3t_1 + t_1t_2,$$

$$\frac{Dv_4}{Bv_3} = -t_1t_2t_3.$$

If we solve these equations for v_1/v_3, v_2/v_3, v_4/v_3, then make them homogeneous, and multiply $v_1 \cdots v_4$ by the common factor A, we shall find

$$\begin{aligned} v_1 &= -2CD + B^2t_1t_2t_3 - BD(t_1 + t_2 + t_3), & v_3 &= AD, \\ v_2 &= -D^2 + 2BCt_1t_2t_3 + BD(t_2t_3 + t_3t_1 + t_1t_2), & v_4 &= -ABt_1t_2t_3. \end{aligned} \tag{21}$$

The coördinates of the planes which osculate the curve in the three points t_1, t_2, t_3, are, according to (16),

$$\begin{aligned} u_1^{(k)} &= B^2t_k^3 - 3BDt_k - 2CD, & u_3^{(k)} &= AD, \\ u_2^{(k)} &= 2BCt_k^3 + 3BDt_k^2 - D^2, & u_4^{(k)} &= -ABt_k^3 \end{aligned} \qquad (k=1, 2, 3).$$

Let $x_1, \cdots x_4$ be the coördinates of the point of intersection of these three planes. We must then have

$$\sum_{i=1}^{4} u_i^{(k)} x_i = 0 \qquad (k=1, 2, 3).$$

Solving these equations we find

$$\begin{aligned} x_1 &= A(t_2t_3 + t_3t_1 + t_1t_2), \\ x_2 &= A(t_1 + t_2 + t_3), \\ x_3 &= 3Bt_1t_2t_3 + 2C(t_2t_3 + t_3t_1 + t_1t_2) + D(t_1 + t_2 + t_3), \\ x_4 &= B(t_2t_3 + t_3t_1 + t_1t_2) + 2C(t_1 + t_2 + t_3) + 3D, \end{aligned} \tag{22}$$

and a simple calculation will show that $\sum v_i x_i = 0$, i. e., as we have stated, the point of intersection of the three osculating planes lies in the plane of the three points of osculation.

In our null-system then, the plane (21) and the point (22) correspond to each other. To find the explicit equations for this correspondence, we need only eliminate t_1, t_2, t_3 between equations (21) and (22). Denoting by ω and ω' two proportionality factors, we find:

$$\begin{aligned} \omega v_1 &= \quad * \quad + 4(C^2 - BD)x_2 + ABx_3 - 2ACx_4, \\ \omega v_2 &= -4(C^2 - BD)x_1 + \quad * \quad + 2ACx_3 - ADx_4, \\ \omega v_3 &= \quad -ABx_1 \quad - \quad 2ACx_2 \quad + \quad * \quad + A^2x_4, \\ \omega v_4 &= \quad 2ACx_1 \quad + \quad ADx_2 \quad - A^2x_3 \quad + \quad *; \end{aligned} \tag{23}$$

$$
\begin{aligned}
\omega' x_1 &= \quad * \quad + A^2 v_2 + \quad AD v_3 \quad + 2AC v_4, \\
\omega' x_2 &= - A^2 v_1 + \quad * \quad - \quad 2AC v_3 \quad - \quad AB v_4, \\
\omega' x_3 &= - AD v_1 + 2AC v_2 + \quad * \quad + 4(C^2 - BD) v_4, \\
\omega' x_4 &= - 2AC v_1 + AB v_2 - 4(C^2 - BD) v_3 + \quad *.
\end{aligned} \tag{24}
$$

But associated with the null-system we have a linear complex, made up of all of the lines passing through a point, which lie at the same time in the plane corresponding to this point in the null-system. Introduce line coördinates by putting $\omega_{ik} = x_i y_k - x_k y_i$, $\omega'_{ik} = v_i w_k - v_k w_i$, where $x_1 \cdots x_4$ and $y_1 \cdots y_4$ are coördinates of two points on the line, and where $v_1 \cdots v_4$, $w_1 \cdots w_4$ are the coördinates of two planes containing the line. *Then the equation of the complex may be written in either of the forms*

$$
-4(C^2 - BD)\omega_{12} - AB\omega_{13} + 2AC\omega_{14} - 2AC\omega_{23} - AD\omega_{42} - A^2\omega_{34} = 0, \tag{25}
$$

$$
\begin{aligned}
&-4(C^2 - BD)\omega'_{34} - AB\omega'_{42} + 2AC\omega'_{23} \\
&\qquad - 2AC\omega'_{14} - AD\omega'_{13} - A^2\omega'_{12} = 0.
\end{aligned} \tag{26}
$$

This complex becomes special if $A^2BD = 0$, i. e., only if the cubic degenerates.

To the flecnode $P_y (x_2 = x_3 = x_4 = 0)$ corresponds the plane

$$
[0, -4(C^2 - BD), -AB, 2AC].
$$

Therefore, if $C^2 - BD = 0$, i. e., *if the derivative cubic is tangent to the generator g, the corresponding plane passes through g.* If $C = 0$, i. e., $u_{11}/u_{22} = \text{const.}$ the plane passes through $P_\sigma P_y$. Therefore, *if the intersections of the cubic with g and the flecnodes form a harmonic group on g, the plane corresponding to each flecnode passes through that point of the derived ruled surface which corresponds to the other.*

§ 3. *The osculating linear complex.*

There is another linear complex associated with every generator of a ruled surface, even more important than the one just considered. A linear complex is determined by five of its lines, provided that these have no two straight line intersectors. Let us consider five generators of a ruled surface, g and four others. As the four other generators approach coincidence with g, a definite linear complex will in general be obtained as a limit. We shall speak of it as the linear complex osculating S along g.

Instead of determining the complex by five consecutive generators of S, it will be advisable to determine it by means of two pairs of lines which are reciprocal polars with respect to it. Two such pairs are obviously constituted by

the flecnode tangent of g and of another generator infinitesimally close to g. Let us denote by g_x, f'_x, f''_x, g'_x respectively the generator of S, the first and the second flecnode tangents and the generator of S' which belong to the argument x. Similarly we denote by $g_{x+\delta x}$, etc., the corresponding lines belonging to the argument $x+\delta x$, where δx is an infinitesimal.

Clearly f'_x and f''_x are the directrices of the osculating linear congruence, (determined by four consecutive generators). Therefore all lines intersecting f'_x and f''_x belong to the osculating linear complex, which must therefore be of the form

$$a\omega_{13} + b\omega_{42} = 0. \tag{27}$$

Change the parameter x by an infinitesimal quantity δx. The flecnode tangents $f'_{x+\delta x}$ and $f''_{x+\delta x}$ must again be the directrices of a linear congruence contained in the complex. We have

$$y_{x+\delta x} = y_x + y'_x \delta x, \qquad \rho_{x+\delta x} = \rho_x + \rho'_x \delta x, \text{ etc.}$$

If we substitute the values of y', ρ', etc., from *Covariants*, equations (21) and (33), we shall find

$$2y_{x+\delta x} = 2y + (\rho - p_{11}y - p_{12}z)\delta x,$$

$$2z_{x+\delta x} = 2z + (\sigma - p_{21}y - p_{22}z)\delta x,$$

$$2\rho_{x+\delta x} = 2\rho + [u_{11}y + u_{12}z - p_{11}\rho - p_{12}\sigma]\,\delta x,$$

$$2\sigma_{x+\delta x} = 2\sigma + [u_{21}y + u_{22}z - p_{21}\rho - p_{22}\sigma]\,\delta x,$$

where of course u_{12} and u_{21} may be equated to zero, since C_y and C_z constitute the flecnode curve.

Now clearly, the coefficients of y, z, ρ, σ in the expressions

$$\rho_{x+\delta x} + \lambda y_{x+\delta x} \qquad \text{and} \qquad \sigma_{x+\delta x} + \mu z_{x+\delta x}$$

will be the coördinates of two arbitrary points P_1 and P_2, situated on $f'_{x+\delta x}$ and $f''_{x+\delta x}$ respectively. If (27) is the equation of the osculating complex, the plane which corresponds in it to P_1 must contain P_2 for arbitrary values of λ and μ. This consideration will enable us to determine the ratio $a:b$.

We find first, remembering that $u_{12} = u_{21} = 0$, for P_1 and P_2 the coördinates:

	x_1	x_2	x_3	x_4
P_1	$2\lambda+(u_{11}-\lambda p_{11})\delta x$	$-\lambda p_{12}\delta x$	$2+(-p_{11}+\lambda)\delta x$	$-p_{12}\delta x$
P_2	$-\mu p_{21}\delta x$	$2\mu+(u_{22}-\mu p_{22})\delta x$	$-p_{21}\delta x$	$2+(-p_{22}+\mu)\delta x$

But, if we denote by $u_1, \cdots, u_4$ the coördinates of the plane which corresponds to the point $x_1, \cdots, x_4$ in the linear complex (27), we find

$$u_1 = -ax_3, \qquad u_2 = +bx_4, \qquad u_3 = ax_1, \qquad u_4 = -bx_2.$$

Substituting for $x_1, \cdots, x_4$ the coördinates of P_1, and writing down the condition that P_2 shall lie in the plane corresponding to P_1, we find that we must have

$$a(\mu - \lambda)p_{21} + b(\lambda - \mu)p_{12} = 0$$

for arbitrary values of λ and μ, i. e., $a : b = p_{12} : p_{21}$.

Therefore, *the equation of the osculating linear complex, in the system of coördinates here employed, is*

$$p_{12}\omega_{13} + p_{21}\omega_{42} = 0. \tag{28}$$

The point-plane correspondence, determined by this complex, is given by the equations

$$u_1 = p_{12}x_3, \qquad u_2 = -p_{21}x_4, \qquad u_3 = -p_{12}x_1, \qquad u_4 = p_{21}x_2. \tag{29}$$

Let us consider a point on the generator g. There will correspond to it, in this complex, a plane, obviously containing the generator itself. But to every point of g there also corresponds another plane through g, viz., the plane tangent to the ruled surface at that point. Clearly, there will exist in general two points on g at which these two planes will coincide. We shall call them the *complex points* of g, and their locus on S, the *complex curve* of the surface. We proceed to determine the complex points of g.

The plane corresponding to any point of g, $(x_1, x_2, 0, 0)$, in the osculating linear complex, has the coördinates

$$u_1 = 0, \qquad u_2 = 0, \qquad u_3 = -p_{12}x_1, \qquad u_4 = p_{21}x_2. \tag{30}$$

The coördinates of the plane, tangent to S at the same point, are found most easily by computing the equation of the plane tangent to the osculating hyperboloid H at that point. They are $(0, 0, x_2, -x_1)$. This plane and (30) coincide if and only if $-p_{12}x_1 : p_{21}x_2 = x_2 : -x_1$, i. e., if

$$p_{12}x_1^2 - p_{21}x_2^2 = 0. \tag{31}$$

This shows that *the complex points and the flecnodes form a harmonic group on every generator of the surface.*

If in (7) we put $u_{11} + u_{22} = 0$, we find that the derivative cubic intersects g precisely in the complex points. Therefore: *if the surface S' of the congruence Γ is so chosen that it intersects the flecnode surface of S in an asymptotic curve, the surface of derivative cubics will intersect S along its complex curve.*

If S is contained in a linear complex, the complex curve is at the same time an asymptotic curve.

For, we notice in the first place that under our assumptions the factors of the the expression $p_{12}z^2 - p_{21}y^2$ determine the complex points. Let us assume $p_{11} - p_{22} = 0$, which we may do without affecting the generality of our argument. Then the condition $\Delta = 0$ for a linear complex becomes $p_{12}/p_{21} = \text{const.}$ If we now make the transformation

$$\eta = \sqrt{p_{21}}\, y + \sqrt{p_{12}}\, z, \qquad \zeta = \sqrt{p_{21}}\, y - \sqrt{p_{12}}\, z,$$

we find that in the transformed system of differential equations, $\pi_{12} = \pi_{21} = 0$, the coefficients of this transformed system being denoted by Greek letters. But this proves that C_η and C_ζ are asymptotic lines on S. It is geometrically evident that the tangents of this asymptotic curve will be lines of the linear complex.

LIE seems to have been the first to notice the existence of this special asymptotic curve on a ruled surface belonging to a linear complex. He noticed in 1871 that its determination requires no integration, and that all other asymptotic curves can be obtained by quadratures.* These latter remarks we can also verify at once from our theory. PICARD found the same theorems independently in 1877.† These theorems on the determination of all of the asymptotic curves by quadratures if one of them is known, follow at once from the fact first noted by BONNET that their equation is of the RICCATI form, and had already been explicitly formulated and applied to special surfaces by CLEBSCH.‡ It seems that VOSS§ was the first to notice that this asymptotic curve and the flecnode curve divide the generators of the surface harmonically. CREMONA‖ however had already observed this in the special case of a surface with two straight line directrices. The general notion of the complex curve, its relation to the derivative surface and to the surface of derivative cubics do not seem to occur in the literature of the subject.

There always exists a pair of points harmonically conjugate to each of two given pairs. We see easily that the pair

$$p_{21}y^2 + p_{12}z^2 \tag{32}$$

is thus situated with respect to the flecnodes as well as the complex points. They are therefore the double points of an involution of which the flecnodes

* LIE, Verhandl. d. Ges. d. Wiss. Christiania (1871), Mathematische Annalen, vol. 5 (1872).

† PICARD, *Thèse*, Paris (1877). See also DARBOUX, Bulletin (1877), p. 335, and Annales de l'Ecole Normale (1877).

‡ CLEBSCH, Crelle's Journal, vol. 68.

§ VOSS, Mathematische Annalen, vol. 8 (1875).

‖ CREMONA, Annali di Matematiche (1867–68).

and the complex points are two pairs. We will call them the *involute points*, their locus the *involute curve*. Of course these three pairs cannot be real at the same time.

Consider the expression

$$\gamma = \theta_4 E - \theta_4' C, \tag{33}$$

where

$$C = u_{12}z^2 - u_{21}y^2 + (u_{11} - u_{22})yz, \qquad E = v_{12}z^2 - v_{21}y^2 + (v_{11} - v_{22})yz.$$

It is easy to prove that γ is a covariant. Moreover it reduces to (32) under our special assumptions. Therefore, *the factors of the covariant* $\theta_4 E - \theta_4' C$ *give the expressions for the involute points in invariant form.* If

$$\theta_4' u_{12} - \theta_4 v_{12} = 0, \qquad \theta_4' u_{21} - \theta_4 v_{21} = 0, \tag{34}$$

the combined locus of C_y and C_z constitutes the involute curve. If however

$$u_{11} - u_{22} = 0, \qquad v_{11} - v_{22} = 0, \tag{35}$$

then C_y and C_z together constitute the complex curve.

We can also write down a covariant whose factors give directly the complex points. It is found by writing down the conditions that a quadratic in y and z shall represent points which divide the pair of the flecnodes, and the pair of involute points harmonically. We find in this way, that the *factors of*

$$\begin{aligned}[(u_{11} - u_{22})v_{12} - (v_{11} - v_{22})u_{12}]z^2 + [(u_{11} - u_{22})v_{21} \\ - (v_{11} - v_{22})u_{21}]y^2 + 2[u_{12}v_{21} - u_{21}v_{12}]yz\end{aligned} \tag{36}$$

represent the complex points. One of the covariants (33) or (36) should be added to the list (*Covariants*, p. 433), which is incomplete as it stands.

We can easily show that the derivative cubic cannot intersect g in its involute points unless S is a quadric. Moreover if the derivative cubic intersects g in two points which are harmonic conjugates with respect to the complex point, S can only be a quadric. The cubic will however intersect g in two points harmonic conjugates with respect to the involute points, provided that $u_{11} + u_{22} = 0$, i. e., provided that S' intersects the flecnode surface of S in an asymptotic curve. It then passes through the complex points.

It is further clear, geometrically as well as analytically, that the two complex points, as well as the two involute points, can coincide only if S has a straight line directrix or if the flecnodes coincide. They become indeterminate if S has two straight line directrices.

To every point P' of g' there corresponds a plane in the osculating linear complex, as well as the plane tangent to S' at P'. When do these planes coincide?

Let the coördinates of P' be $(0, 0, \alpha_1, \alpha_2)$. The plane, corresponding to P' in the complex, has the coördinates $(\alpha_1 p_{12}, -\alpha_2 p_{21}, 0, 0)$, so that it contains g'. The coördinates of the plane tangent to S' at P' are of course the same as those of the plane tangent to H' at P', which may be obtained from (5), viz: $(u_{11} u_{22}^2 \alpha_2, -u_{11}^2 u_{22} \alpha_1, 0, 0)$. These planes coincide if and only if

$$u_{11} p_{12} \alpha_1^2 = u_{22} p_{21} \alpha_2^2.$$

The corresponding points on g are again harmonic conjugates with respect to the flecnodes, i. e., those asymptotic tangents of S which join g to the points of g', the planes corresponding to which in the linear complex are the planes tangent to S', are harmonic conjugates with respect to the flecnode tangents.

The planes which correspond to these two points of g' in the null-system of the cubic, do not contain g'.

§4. *Relation of the osculating linear complex to the linear complex of the derivative cubic.*

The equations of the two complexes are

$$\begin{aligned} \Omega_1 &= 4(C^2 - BD)\omega_{12} + AB\omega_{13} + AD\omega_{42} \\ &\qquad + A^2\omega_{34} + 2AC\omega_{23} - 2AC\omega_{14} = 0, \\ \Omega_2 &= p_{12}\omega_{13} + p_{21}\omega_{42} = 0. \end{aligned} \tag{37}$$

Their simultaneous invariant is

$$-2u_{11}u_{22}(u_{11} - u_{22})^2(u_{11} + u_{22})p_{12}p_{21}, \tag{38}$$

which, leaving aside the cases when S' is developable or when S has one or more straight line directrices, vanishes if and only if $u_{11} + u_{22} = 0$. Therefore, *the osculating linear complex and the complex of the derivative cubic are in involution if the first derivative ruled surface cuts out asymptotic curves on the flecnode surface of S, and the cubic passes through the complex points of g.* Some of our previous theorems are consequences of this.

The two special complexes which are contained in the family

$$\lambda\Omega_1 + \mu\Omega_2 = 0,$$

where λ and μ are constants, are those for which

$$-3A^2BD\lambda^2 - 2u_{11}u_{22}(u_{11} - u_{22})^2(u_{11} + u_{22})p_{12}p_{21}\lambda\mu + p_{12}p_{21}\mu^2 = 0,$$

or, discarding again the case when S has a straight directrix,

$$-12u_{11}^3u_{22}^3(u_{11} - u_{22})^4\lambda^2 - 2u_{11}u_{22}(u_{11} - u_{22})^2(u_{11} + u_{22})\lambda\mu + \mu^2 = 0.$$

They coincide if

$$u_{11}^2u_{22}^2(u_{11} - u_{22})^4(u_{11} + u_{22})^2 + 12u_{11}^3u_{22}^3(u_{11} - u_{22})^4 = 0,$$

i. e, if S' is developable, if S is a quadric, or if

$$(u_{11} + u_{22})^2 + 12u_{11}u_{22} = 0. \tag{39}$$

We can always choose the independent variable so as to satisfy this condition. In fact, if we change the independent variable by putting $\xi = \xi(x)$, according to *Congruence*, equation (5), we shall have

$$(\bar{u}_{11} + \bar{u}_{22})^2 + 12\bar{u}_{11}\bar{u}_{22} = 0,$$

if ξ be taken as any solution of the equation

$$64\mu^2 + 32(u_{11} + u_{22})\mu + (u_{11} + u_{22})^2 + 12u_{11}u_{22} = 0, \tag{40}$$

where

$$\mu = \{\xi, x\} = \eta' - \tfrac{1}{2}\eta^2, \qquad \eta = \frac{\xi''}{\xi'}.$$

Therefore, there exist two families of ∞^1 non-developable ruled surfaces in the congruence Γ such that the linear congruence, common to the osculating linear complex of S and the linear complex of the derivative cubic, shall have coincident directrices. Any four surfaces of one family intersect all of the asymptotic tangents of S in a point row of constant anharmonic ratio. The two families never coincide unless $\theta_4 = 0$, i. e., unless the flecnode curve intersects every generator in two coincident points. But in this case the congruence is not defined. If S has a straight line directrix this congruence is degenerate.

The coördinates of the plane, which corresponds to a point $(x_1, x_2, 0, 0)$ of g in the null-system of the cubic, are

$$4(C^2 - BD)x_2, \quad -4(C^2 - BD)x_1, \quad -ABx_1 - 2ACx_2, \quad 2ACx_1 + ADx_2.$$

This plane contains g if and only if $C^2 - BD = 0$, i. e., if the derivative cubic is tangent to g. It will coincide with the plane tangent to S at this point, if further

$$-ABx_1 - 2ACx_2 = \omega x_2, \qquad 2ACx_1 + ADx_2 = -\omega x_1,$$

where ω is a proportionality factor, or

$$ABx_1 + (2AC + \omega)x_2 = 0, \qquad (2AC + \omega)x_1 + ADx_2 = 0,$$

whence follows $\omega = -AC$ or $-3AC$. We have therefore

$$x_1 : x_2 = -C : B = -D : C \qquad \text{or} \qquad x_1 : x_2 = C : B = D : C.$$

These points are harmonic conjugates with respect to the flecnodes.

Therefore, if the derivative cubic is tangent to g, there are two points of g whose tangent planes are the planes corresponding to them in the null-system of the cubic. These points and the flecnodes form a harmonic group on g. They never coincide with the complex points unless the ruled surface has a straight line directrix.

The planes, corresponding to a point of g in the null-system of the cubic and in the osculating complex, coincide if

$$C^2 - BD = 0,$$
$$(AB - \omega p_{12})x_1 + 2ACx_2 = 0,$$
$$2ACx_1 + (AD - \omega p_{21})x_2 = 0,$$

where ω is a root of the quadratic

$$\omega^2 + 2u_{11}u_{22}(u_{11} - u_{22})^2(u_{11} + u_{22})\omega - 12u_{11}^3u_{22}^3(u_{11} - u_{22})^4 = 0, \tag{41}$$

neglecting again the case when S has a straight line directrix. These two points of g coincide if $(u_{11} + u_{22})^2 + 12u_{11}u_{22} = 0$.

More generally, if we write down the conditions that the same plane shall correspond to a point (x_1, x_2, x_3, x_4) in the osculating linear complex and in the complex of the cubic, we shall obtain as the locus of these points two straight lines, the directrices of the congruence common to the two complexes. These conditions are as follows, $x_1 \cdots x_4$ must satisfy the equations:

$$\begin{aligned}
* \; + 4(C^2 - BD)x_2 + (AB - \omega p_{12})x_3 - 2ACx_4 &= 0,\\
-4(C^2 - BD)x_1 \; + \; * \; + 2ACx_3 - (AD - \omega p_{21})x_4 &= 0,\\
-(AB - \omega p_{12})x_1 - 2ACx_2 \; + \; * \; + A^2x_4 &= 0,\\
2ACx_1 + (AD - \omega p_{21})x_2 - A^2x_3 \; + \; * \; &= 0,
\end{aligned} \tag{42}$$

the vanishing of whose skew-symmetric determinant gives for ω the quadratic equation (41), which may also be written

$$(AB - \omega p_{12})(AD - \omega p_{21}) + 4A^2(C^2 - BD) - 4A^2C^2 = 0. \tag{41a}$$

Let ω_1 and ω_2 be the two roots of this equation. If we eliminate x_3 from the first two, x_4 from the last two equations of (42), if we make use of (41a) and assume that neither A nor $C^2 - BD$ is zero, we shall find

$$\begin{aligned}
-(AB - \omega_k p_{12})x_1 - 2ACx_2 + A^2x_4 &= 0,\\
-4(C^2 - BD)x_2 - (AB - \omega_k p_{12})x_3 + 2ACx_4 &= 0
\end{aligned} \qquad (k = 1, 2),$$

whence

$$\begin{aligned}
-2C(AB - \omega_k p_{12})x_1 - 4ABDx_2 + A(AB - \omega_k p_{12})x_3 &= 0,\\
-(AB - \omega_k p_{12})x_1 - 2ACx_2 + A^2x_4 &= 0
\end{aligned} \qquad (k = 1, 2), \tag{43}$$

the equations of the two directrices in simpler form than in (42).

A line joining the point $(x_1, x_2, 0, 0)$ of g to the point $(0, 0, x_1, x_2)$ of g' is a generator of the second kind on H. It is not difficult to see that it will intersect the directrix (43) if and only if

$$-(AB - \omega_k p_{12})^2x_1^2 + 4A^2BDx_2^2 = 0. \tag{44}$$

Hence, the two points, in which either of the directrices of the congruence common to the two complexes intersects the osculating hyperboloid, determine upon this hyperboloid two generators of the second set which are harmonic conjugates with respect to the flecnode tangents.

It also follows easily that the two pairs thus obtained, one corresponding to each directrix, coincide only if $(u_{11}+u_{22})^2+12u_{11}u_{22}=0$, i. e., if the directrices themselves coincide. Further if one of these pairs intersects g in the involute points, the same is true of the other pair, so that this can only happen if the directrices coincide. Finally, such a pair of generators of H can pass through the complex points only if S has a straight line directrix, or if S' is developable.

The line joining the points $(x_1, 0, x_3, 0)$ and $(0, x_1, 0, x_3)$ is a generator of the first set on H. The coördinates of an arbitrary point of this line are $(\lambda x_1, \mu x_1, \lambda x_3, \mu x_3)$. This line will therefore intersect one of the directrices of the congruence if x_1, x_3, λ, μ can be determined so as to satisfy the equations

$$\lambda[-2C(AB-\omega_k p_{12})x_1+A(AB-\omega_k p_{12})x_3]-4\mu ABDx_1=0,$$
$$-\lambda(AB-\omega_k p_{12})x_1+\mu(-2ACx_1+A^2x_3)=0,$$

which gives either $A=0$, $AB-\omega_k p_{12}=0$, or

$$4(C^2-BD)x_1^2-4ACx_1x_3+A^2x_3^2=0. \tag{45}$$

The first two cases give either a surface S with a straight line directrix, or else a developable surface S'. Leaving these cases aside we notice that (45) does not contain ω_k so that if the line on H here considered intersects one of the directrices it intersects the other also. Combining this with our previous result, we see that the following theorem holds.

The four points in which the directrices of the congruence, common to the osculating linear complex and the linear complex of the derivative cubic, intersect the osculating hyperboloid can be grouped into two pairs, such that the line joining the members of each pair shall be a generator of the first set upon the hyperboloid. Upon this generator this pair of points, together with the intersections of the generator with the flecnode tangents, form a harmonic group.

The plane, corresponding to a point $(x_1, x_2, 0, 0)$ of g in the null-system of the cubic, intersects the flecnode tangents

$$f' \text{ in the point } [ABx_1+2ACx_2, 0, 4(C^2-BD)x_2, 0],$$
$$f''' \text{ in the point } [0, 2ACx_1+ADx_2, 0, 4(C^2-BD)x_1].$$

The line joining these points is a generator of H, if either

$$A=0, \quad \text{or} \quad C^2-BD=0, \quad \text{or} \quad Bx_1^2-Dx_2^2=0.$$

Therefore, *there exist in general two points on g, harmonic conjugates with*

respect to the flecnodes, such that the planes, corresponding to them in the null-system of the derivative cubic, pass through a generator of the osculating hyperboloid. If the cubic is tangent to g the null-plane of *any point* of g contains a generator of H, viz. g itself. If $A = 0$ likewise, *all points* of g satisfy the condition of the theorem. Their null-planes all pass through g'.

§ 5. *Various theorems concerning the flecnode surface. The principal surface of the congruence* Γ.

Let us consider the planes which osculate the flecnode curve of S at P_y and P_z. We have of course $u_{12} = u_{21} = 0$. If x_1, x_2, x_3, x_4 are the coördinates of an arbitrary point of the plane osculating C_y at P_y, we have for the equation of this plane

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \end{vmatrix} = 0.$$

But

$$2y_k' = \rho_k - p_{11}y_k - p_{12}z_k, \qquad 2z_k' = \sigma_k - p_{21}y_k - p_{22}z_k,$$

$$-y_k'' = p_{11}y_k' + p_{12}z_k' + q_{11}y_k + q_{12}z_k, \text{ etc.}$$

If we assume again $p_{11} = p_{22} = 0$, and substitute into the above equation, it becomes

$$\begin{vmatrix} x_1, & x_2, & \cdots \\ y_1, & y_2, & \cdots \\ \rho_1 - p_{12}z_1, & \rho_2 - p_{12}z_2, & \cdots \\ p_{12}\sigma_1 + 2q_{12}z_1, & p_{12}\sigma_2 + 2q_{12}z_2, & \cdots \end{vmatrix} = 0.$$

But if we introduce again our fundamental tetrahedron $P_y P_z P_\rho P_\sigma$, this becomes

$$p_{12}x_2 + p_{12}^2 x_3 - 2q_{12}x_4 = 0. \tag{46a}$$

In the same way we find the equation of the plane osculating C_z at P_z to be

$$p_{21}x_1 - 2q_{21}x_3 + p_{21}^2 x_4 = 0. \tag{46b}$$

To these equations must be added the conditions $u_{12} = u_{21} = 0$, if C_y and C_z are the two portions of the flecnode curve.

Let us assume that this is the case. The osculating planes at P_y and P_z intersect along a straight line, whose intersections with the osculating hyperboloid may now be found. If $x_1, \cdots, x_4$ are the coördinates of one of these points of intersection, we find

$$\begin{aligned} p_{12}p_{21}^2 x_2 &= -2q_{12}p_{21}x_1 - (p_{12}^2 p_{21}^2 - 4q_{12}q_{21})x_3, \\ p_{12}p_{21}^2 x_4 &= -p_{12}p_{21}x_1 + 2q_{21}p_{12}x_3, \end{aligned} \tag{47}$$

where the ratio of $x_1 : x_3$ is determined by the quadratic

$$(48) \qquad (p_{12}^2 p_{21}^2 - 4q_{12}q_{21})x_3^2 + 2(p_{12}q_{21} + p_{21}q_{12})x_1x_3 - p_{12}p_{21}x_1^2 = 0.$$

Therefore if $p_{12}q_{21} + p_{21}q_{12} = 0$, i. e., *if* $p_{12}p_{21} = const.$, *the two generators of the first kind on* H *which pass through these points are harmonic conjugates with respect to* g *and* g'. *If*

$$(49) \qquad (p_{12}q_{21} - p_{21}q_{12})^2 + p_{12}^3 p_{21}^3 = 0,$$

the intersection of the two osculating planes is tangent to the hyperboloid. This latter property is obviously characteristic of a class of ruled surfaces, and can be expressed in invariant form.

If $\theta_4^3\theta_9^2 + 16\theta_{10}^3 = 0$, *the ruled surface* S *has the following characteristic property. The planes, which osculate the flecnode curve at the two points of its intersection with any generator, intersect in a line which is tangent to the osculating hyperboloid.*

To prove this it is sufficient to notice that the invariant equation

$$(49a) \qquad \theta_4^3\theta_9^2 + 16\theta_{10}^3 = 0$$

reduces to (49) under our special assumptions. Further we notice:

If the independent variable is chosen so that $\theta_{10}/\theta_4 = const.$, *the generator* g' *of the first derived surface is the harmonic conjugate of* g *with respect to the two generators of the same kind on* H *which are determined by the points in which the line of intersection of the two osculating planes intersects* H.

The plane osculating the flecnode curve at P_y intersects the flecnode tangent f''' which passes through P_z in the point $(0, 2q_{12}, 0, p_{12})$. Similarly, the plane osculating C_z at P_z intersects f' in the point $(2q_{21}, 0, p_{21}, 0)$. Therefore, the line joining these points is a generator of H if and only if $p_{12}q_{21} - p_{21}q_{12} = 0$, i. e., if S belongs to a linear complex. In other words: *the points in which the two planes, osculating the flecnode curve at its points of intersection with any generator, intersect the flecnode tangents are situated upon the same generator of the osculating hyperboloid if and only if the surface belongs to a linear complex.*

To each of the two planes (46a) and (46b) corresponds a point in that plane by means of the osculating linear complex. These points have the coördinates

$$(-p_{12}p_{21}, -2q_{12}, 0, -p_{12}) \qquad \text{and} \qquad (2q_{21}, p_{12}p_{21}, p_{21}, 0).$$

We find that the line joining them intersects H in two points which form a harmonic group with the first two, if S belongs to a linear complex. It is tangent to H if (49) is satisfied. *Therefore, if the two planes, osculating the flecnode curve at its two points of intersection with a generator, intersect in a line which is tangent to the osculating hyperboloid, the line joining the two points of these planes which correspond to them in the osculating linear complex is also tangent to the osculating hyperboloid, and conversely.*

We have seen that, under the assumptions $u_{12} = u_{21} = p_{11} = p_{22} = 0$, the equations of the sheet F' of the flecnode surface assume the form (15). Let us denote by $\mathfrak{U}_{ik}$ the quantities formed for this system according to the same law as are the quantities u_{ik} for the equations of S. Then we shall have

$$\mathfrak{U}_{12} = 0, \quad \mathfrak{U}_{21} = -4(q'_{11} - q'_{22}) + 8\frac{(q_{11} - q_{22})q_{12}}{p_{12}}, \quad \mathfrak{U}_{11} - \mathfrak{U}_{22} = 4(q_{11} - q_{22}).$$

The curve C_y is a branch of the flecnode curve on F' as well as on S. The other branch is the locus of the point

$$\mathfrak{U}_{21}y - (\mathfrak{U}_{11} - \mathfrak{U}_{22})\rho.$$

Now if the transformation $\xi = \xi(x)$ is made, ρ is converted into

$$\bar{\rho} = \frac{1}{\xi'}(\rho + \eta y), \qquad \text{where} \qquad \eta = \frac{\xi''}{\xi'}.$$

Therefore, if a transformation $\xi_1 = \xi_1(x)$ is made such that the derivative surface of S with respect to ξ_1 shall cut out upon F' the second branch of its flecnode curve, ξ_1 must be so chosen that

$$\eta_1 = \frac{q'_{11} - q'_{22}}{q_{11} - q_{22}} - 2\frac{q_{12}}{p_{12}}. \tag{50a}$$

Similarly the second branch of the flecnode curve on F'' will be obtained by putting

$$\eta_2 = \frac{q'_{11} - q'_{22}}{q_{11} - q_{22}} - 2\frac{q_{21}}{p_{21}}. \tag{50b}$$

The two surfaces of Γ thus obtained coincide only if S belongs to a linear complex, i. e., *the second branches of the flecnode curves on the two sheets of the flecnode surface of S correspond to each other only if S belongs to a linear complex.*

We have seen that the plane osculating C_y at P_y intersects f'' in the point $2q_{12}z + p_{12}\sigma$. The corresponding point on f', i. e., the point obtained by finding the intersection of f' with the corresponding generator of H, is given by $2q_{12}y + p_{12}\rho$. We find a surface of Γ which intersects f' and f'' in these points by making a transformation of the independent variables for which $\eta = 2q_{12}/p_{12}$, or if we denote this special value of η by η_1,

$$\bar{\eta}_1 = 2\frac{q_{12}}{p_{12}}.$$

If we now denote by η the expression

$$\eta = \tfrac{1}{2}\frac{q'_{11} - q'_{22}}{q_{11} - q_{22}},$$

we find

$$\eta = \tfrac{1}{2}(\eta_1 + \bar{\eta}_1),$$

and similarly

$$\eta = \tfrac{1}{2}(\eta_2 + \bar{\eta}_2).$$

This gives us a very important result. For, we have frequently made use of a normal form for our system of differential equations, in which $\theta_4 = \text{const.}$ But in order to have $\theta_4 = \text{const.}$, we must make precisely the transformation determined by η. On account of its importance, we will call the surface of Γ which is thus obtained, the *principal surface* of the congruence, and the curves in which it intersects the two sheets of the flecnode surface of S their *principal curves*. We see then that the principal surface may be constructed as follows.

We consider the flecnodes P_y and P_z of g, the planes p_y and p_z osculating the flecnode curve at these points, and the points P' and P'' upon the flecnode tangents f' and f''' whose loci are the second branches of the flecnode curves on the two sheets F' and F''' of the flecnode surface. The plane p_y intersects f''' in a certain point to which corresponds a point on f' such that the line joining them is a generator of the osculating hyperboloid. This latter point together with P' constitute a pair, such that the harmonic conjugate of P_y with respect to it is the point in which the principal surface intersects f'. The intersection with f''' is found in the same way.

We might also say, that in this way there is determined, upon the generators of H, an involution whose double elements are g and the generator of the principal surface.

This construction is especially important for the complete geometrical interpretation of the covariant C_3. For, the interpretation given in a former paper, (*Covariants*, p. 450), rests upon the use of the principal surface, since θ_4 is assumed to be constant, a fact which appears clearly from the detailed proof of the theorem there given, but which is not mentioned explicitly in its enunciations as it should have been.

By combining a number of our previous results with the notion of the principal surface, we obtain a number of theorems, which may be easily verified. They provide interpretations for the vanishing of certain invariants, and therefore furnish characteristic properties of certain families of ruled surfaces. I write them down without proof.

If $\theta_{15} = 0$, $\theta_4 \neq 0$, the principal surface is the harmonic conjugate of S with respect to the two ruled surfaces of Γ which cut out the second branches of the flecnode curves on F' and F'''.

If $\theta_6\theta_4 - 9\theta_{10} = 0$, *the principal surface intersects F' and F'' along asymptotic curves.*

If $4\theta_4^3 - (\theta_4\theta_6 - 9\theta_{10})^2 = 0$, the principal surface is developable.

If $\theta_{15}^2 - \theta_9^2\theta_4^3 = 0$, the principal surface intersects one of the sheets of the flecnode surface along the second branch of its flecnode curve. It thus intersects both sheets if $\theta_{15} = \theta_9 = 0$.

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